A THEOREM ON OVERCONVERGENCE

F. SUNYER I BALAGUER

The conjecture announced by A. J. Macintyre \[2; 3\] is equivalent to the theorem stated and proved below.

**Theorem.** Let \( D \) be an open domain containing the origin and let \( f(z) \) be a function regular in \( D \) with the expansion \( f(z) = \sum_{n=0}^{\infty} c_n z^n \). Let \( D_1 \) be a bounded closed domain contained in \( D \). Then there exists a positive number \( \lambda_0 = \lambda_0(D, D_1) \) such that if \( c_n = 0 \) for a sequence of intervals \( n_k \uparrow n \geq \lambda n_k \) with \( \lambda > \lambda_0 \), then the subsequence of partial sums \( s_{n_k} = \sum_{n=0}^{n_k} c_n z^n \) converges uniformly to \( f(z) \) in \( D_1 \).

**Proof.** Let \( CD \) and \( CD_1 \) denote the complements of \( D \) and \( D_1 \) respectively and let \( h_i, i = 1, 2, \ldots \), be the components of \( CD_1 \). The components can be considered as disjoint and there exists only one unbounded component. The one unbounded component will be denoted as \( h_1 \).

One can assert that there exists only a finite number of components \( h_i \) such that

\[ h_i \cap CD \neq \emptyset, \]

where \( \emptyset \) is the empty set. This assertion is proved as follows. Assume that there exists an infinite number of components \( h_i, i \geq 2 \), such that (1) is valid. A bounded sequence of points \( a_i \) can be formed where \( a_i \in h_i \cap CD, i \geq 2 \). Every \( a_i \) is an element of \( CD \) and hence the dis-
tance $d$ from $D_1$ is at least $\delta > 0$. The limit point $a$ of the sequence then must be such that $d(a, D_i) \geq \delta > 0$. Thus $a$ is an element of $CD_1$ and all points $z$ in $|z-a| < \delta$ must be in the same component.

Let the finite number of components be enumerated as $h_i, i = 1, 2, \cdots, N$. Considering now

$$D_2 = D_1 + \bigcup_{i=N+1}^{\infty} h_i,$$

then $D_2$ is a bounded closed domain and $D_2 \subset D_1$. Since $\sum_{i=N+1}^{\infty} h_i$ is bounded and $D_1$ is bounded by hypothesis, $D_2$ is bounded. Also, since $h_i \cap CD = \emptyset, i \geq N+1$, then $h_i \subset D$ and $D_2$ is contained in $D$. To prove that $D_2$ is closed note that its complement is $\bigcup_{i=N+1}^{\infty} h_i$ and is open.

Now $N-1$ polygonal arcs $L_1, L_2, \cdots, L_{N-1}$ can be chosen such that $D_2 - \sum_{i=N+1}^{N-1} L_k$ is simply connected. Also $N-1$ other polygon arcs $L'_1, L'_2, \cdots, L'_{N-1}$ can be so chosen that $L_k \cap L'_i = \emptyset$ and $D_2 - \sum_{i=N+1}^{N-1} L'_k$ is simply connected. Consider now the open circle $C(s, R)$ or $|z-s| < R$ and let $S(L, R) = \bigcup_{L \subset C(s, R)}$. Thus $S(L, R)$ is a strip enclosing the polygonal arc $L$. For $R$ sufficiently small,

$$S(L_k, R) \cap S(L'_j, R) = \emptyset.$$ 

Hence for $R$ sufficiently small two closed simply connected domains can be defined, $D_3 = D_2 - \bigcup_{1}^{N-1} S(L_k, R)$ and $D'_3 = D_2 - \bigcup_{1}^{N-1} S(L'_j, R)$ such that $D_3 + D'_3 = D_2$. This follows from

$$D_3 + D'_3 = D_2 - \left\{ \bigcup_{k=1}^{N-1} S(L_k, R) \right\} \cap \left\{ \bigcup_{j=1}^{N-1} S(L'_j, R) \right\} = D_2$$

by (3).

The proof of the theorem follows. An open bounded simply connected domain $\Delta = \Delta(D, D_3)$ can now be defined such that $D_3 \subset \Delta$, $\{ |z| < r \} \subset \Delta$, $\bar{\Delta} \subset D$ where $r$ is the radius of convergence of $f(z) = \sum_{n} c_n z^n$ and $\bar{\Delta}$ is the closure of $\Delta$. From the Nevanlinna two-constant theorem, if $F(z)$ is regular in $\Delta$

$$M(\Delta) = \lim\sup_{z \to \Delta} |F(z)|, \quad M(d) = \lim\sup_{|z| < r/2} |F(z)|,$$

then [1]

$$M(D_3) = \lim\sup_{z \to D_3} |F(z)| \leq \left\{ M(\Delta) \right\}^\theta \left\{ M(d) \right\}^{1-\theta}$$

where $\theta > 0$ depends on $D_3$ and $\Delta$. Using the majorization of $r_n$, where $r_n = f(z) - s_n, f_n = \sum_{n} c_n z^n$, if $n_k$ is large we get $\lim\sup_{|z| < r/2} |r_{n_k}| < \ldots
$H(3/4)^{hn_k}$ and $\text{l.u.b.}_{z \in \Delta} |r_{nk}| < H_1^{n_k}$ where $H$ and $H_1$ are two constants depending only on $\Delta$. Thus by (4),

$$\text{l.u.b.}_{z \in D_\delta} |r_{nk}| \leq H^{1-\theta} \{ H_1^\theta (3/4)^{\lambda(1-\theta)} \}^{n_k}.$$  

Thus if $\lambda > \lambda_0(\Delta, D_\delta)$ there is overconvergence in $D_\delta$. Similarly there is overconvergence in $D_{\delta'}$ if $\lambda > \lambda_0(\Delta, D_{\delta'}')$. Now since $D_\delta + D_{\delta'} = D_2 \supset D_1$ the theorem is proved.

**Remark.** By the same method similar results are proved for the series of Dirichlet and for the integral of Laplace.

**References**


**University of Barcelona, Spain**