ON BOUNDED FUNCTIONS WITH ALMOST PERIODIC DIFFERENCES

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The aim of this paper is to generalize to groups the well-known Bohl-Bohr theorem, which states that if the indefinite integral, $F(x) = \int_0^x f(t) \, dt$, of an almost periodic function $f(x)$, is bounded, then it is almost periodic (see [1] or [2]).

No expression of the form $\int_a^x f(t) \, dt$ is available in groups, but observing that $F(a+x) - F(x) = \int_a^x f(t) \, dt$ is easily proved to be almost periodic, whatever be the constant $a$, we are led to the following

THEOREM. Let $G$ be a multiplicative group and let the left differences $F(ax) - F(x)$ be right almost periodic for every $a \in G$, where $F$ is a given complex-valued function on $G$. If $F(x)$ is bounded then it is right almost periodic.

We recall that a real or complex function $\phi(x)$ is right almost periodic if, from every sequence $(c_n)$ we can extract a subsequence $(b_n)$ for which the functions $\phi(xb_n)$ converge uniformly in $G$. In that case to every $\epsilon > 0$ there corresponds a finite number of elements of $G$, say $s_1, \ldots, s_k$, such that to every $t \in G$ we can associate an integer $i \leq k$ for which

$$| \phi(xt) - \phi(xs_i) | < \epsilon,$$

whatever be $x \in G$.

Proof of the theorem. It is sufficient to consider the case of a real function. Suppose that $F(x)$ is not right almost periodic. Then there exists an $\alpha > 0$ and a sequence $(c_n)$, such that, in every subse-
quence \((b_n) \subset (c_n)\) we can find \(b_p, b_q\) for which \(\sup |F(xb_p) - F(xb_q)| > \alpha\). We can even suppress the modulus sign by exchanging if necessary \(b_p\) and \(b_q\). In other words there exist \(t \in G, b_p, b_q\) such that

\[
F(t b_q^{-1} b_p) - F(t) > \alpha.
\]

We shall prove that if \(F(a_1 x_1) - F(x_1) > \beta\), we can find \(x_2, a_2 \in G\) such that \(F(a_2 x_2) - F(x_2) > \beta + \alpha\). This will show that \(F(x)\) is unbounded, against the hypothesis, and the theorem will be proved.

So put \(\phi(x) = F(a_1 x) - F(x)\) and suppose that

\[
(2) \quad \phi(x_1) = \beta + \epsilon > \beta.
\]

Since \(\phi(x)\) is right almost periodic let \(s_1, \ldots, s_k\) be such that to every \(t \in G\) we can associate an integer \(i \leq k\) for which \(|\phi(xt) - \phi(xs_i)| < \frac{\epsilon}{2}\).

In particular, for \(x = x_1 s_i^{-1}\): \(|\phi(x_1 s_i^{-1}t) - \phi(x_1)| < \frac{\epsilon}{2}\), so that by (2) \(\phi(x_1 s_i^{-1}t) > \beta + \epsilon/2\), i.e.,

\[
(3) \quad F(a_1 x_1 s_i^{-1}t) - F(x_1 s_i^{-1}t) > \beta + \epsilon/2.
\]

Now consider the right almost periodic functions

\[
\phi_i(x) = F(a_1 x s_i^{-1}x) - F(x) \quad (i = 1, \ldots, k).
\]

We can extract from the sequence \((c_n)\) a subsequence \((b_n)\) such that \(|\phi_i(xb_q^{-1} b_p) - \phi_i(x)| < \frac{\epsilon}{2}\), whatever be \(x \in G, b_p, b_q,\) and \(i = 1, \ldots, k\).

We deduce

\[
| F(a_1 x_1 s_i^{-1} t b_q^{-1} b_p) - F(t b_q^{-1} b_p) - F(a_1 x_1 s_i^{-1} t) + F(t) | = | \phi_i(t b_q^{-1} b_p) - \phi_i(t) | < \frac{\epsilon}{2}.
\]

Hence by (1)

\[
(4) \quad F(a_1 x_1 s_i^{-1} t b_q^{-1} b_p) - F(a_1 x_1 s_i^{-1} t) > \alpha - \epsilon/2.
\]

(3) and (4) give, by addition, the required relation:

\[
F(a_1 x_1 s_i^{-1} t b_q^{-1} b_p) - F(x_1 s_i^{-1} t) > \alpha + \beta.
\]

The proof is now complete.

References


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