ON COMPACT KAEHLERIAN MANIFOLDS WITH POSITIVE HOLOMORPHIC CURVATURE

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1. Statement of the results.

1.1. Let M be a compact Kaehlerian manifold. The underlying Riemannian manifold which we also denote by M is orientable and of even dimension. Let $K=K(\sigma)$ be the Riemannian curvature of M, considered as a Riemannian manifold. $K(\sigma)$ is a function on the 2-planes σ tangent to M. The restriction of K to holomorphic 2-planes is called holomorphic curvature and will be denoted by hol K.

In this paper we consider Kaehlerian manifolds with strictly positive holomorphic curvature hol K. Since M is compact, hol K has a maximum on M. We assume the Kaehler metric normed in such a way that the maximum of hol K is $\leqq 1$. That is to say, we assume that hol K satisfies the inequalities

$$0 < \text{hol } K \leqq 1. \tag{1}$$

Berger [2] has shown, that the inequalities (1) for the holomorphic curvature hol K imply $K=K(\sigma)\leqq 1$ for the Riemannian curvature K, with $K(\sigma)=\max \text{hol } K$ for holomorphic 2-planes σ only. Note that (1) does not imply $0<K(\sigma)$ for all 2-planes σ.

1.2. Using a similar argument as in the proof of Theorem 1 in the paper [4], we prove the following

THEOREM 1. *Let M be a compact Kaehlerian manifold satisfying* (1). *Then the following holds*:

(i) *M is simply connected.*

(ii) *The length of a closed geodesic on M, considered as Riemannian manifold is at least 2π.*

(iii) *If two points of M have a distance $<\pi$, then there is a unique geodesic segment of length $d(p, q)$ joining them, where $d(p, q)$ is the distance between p and q. That is to say, the distance between a point and its cut locus is $\geqq\pi$.*

(iv) *The diameter of M, i.e., the maximal distance two points of M can have, is at least π.*

(v) (*Tsukamoto* [7].) *On the other hand, if H is a positive lower bound for* hol K *on M, then $\pi/H^{1/2}$ is an upper bound for the diameter of M.*

REMARKS. 1. The cut locus $C(p)$ of a point p of a compact Riemannian manifold is the complement of the greatest open set of points

Received by the editors July 13, 1959, and in revised form, May 19, 1960.

q of M, which can be joined to p by an unique geodesic segment of minimal length $d(p, q)$. For more information on the cut locus we refer to [4] and the references given there.

2. There holds an even stronger statement than (ii): *If* $0 < \text{hol}\, K \leqq 1$, *then the length* $L(G)$ *of a geodesic loop* G *is* $\geqq 2\pi$; *and* $L(G) = 2\pi$ *can hold only if* G *is a closed geodesic.* This follows from an argument used in the proof of Proposition 2, cf. §3.3 below.

3. The lower bounds given in (ii), (iii), (iv) are the best possible as can be seen from the complex projective space $P(\mathbf{C})$ with the usual Kaehlerian metric of constant holomorphic curvature equal to 1; cf. Bochner [3].

4. Combining (iv) and (v) we have: *If the holomorphic curvature* hol K *of a compact Kaehlerian manifold satisfies the inequalities* $0 < H \leqq \text{hol}\, K \leqq 1$, *then the diameter* $d(M)$ *of* M *satisfies the inequalities*

$$\pi \leqq d(M) \leqq \pi/H^{1/2}. \tag{2}$$

Here the first inequality becomes an equality if M is the complex projective space $P(\mathbf{C})$ with the usual metric. There naturally arises the question whether this is the only space satisfying (1) and $d(M) = \pi$. We will answer this question affirmatively, see Theorem 2 below. The corresponding question whether $P(\mathbf{C})$ is also the only space satisfying $0 < H \leqq \text{hol}\, K$ and $d(M) = \pi/H^{1/2}$ remains open. Only in the special case that M has real dimension 2, i.e., that M is homeomorphic to the 2-sphere, has this question been answered affirmatively; see Toponogov [6] and, for a brief sketch of the proof, [5].

1.3. THEOREM 2. *Let* M *be a compact Kaehlerian manifold satisfying* (1). *If the diameter* $d(M)$ *of* M *is equal to* π, *then* M *is the complex projective space with the usual Kaehlerian structure of constant holomorphic curvature* hol $K = 1$.

REMARKS. 1. If M has real dimension 2, i.e., if M is homeomorphic to the 2-sphere, this theorem is contained in the Theorem 2 of the paper [5].

2. In the proof of Theorem 2 the following lemma plays an essential role. The proof of this lemma is contained in the proof of the Theorem 1 of the paper [4]:

LEMMA. *Let* M *be a compact, orientable Riemannian manifold of even dimension and of positive Riemannian curvature* $K \leqq 1$. *Let* p *and* q *be two points of* M *which are joined by two geodesic segments* Γ *and* Δ *of length* π *and such that* Γ *and* Δ *start at* p *in opposite directions and arrive again at* q *in opposite directions*; Γ *and* Δ *thus form a closed geodesic of length* 2π. *Then, for each angle* α, $0 \leqq \alpha \leqq \pi$, *there exists a geo-*

desic segment $\Gamma(\alpha)$ *of length* π *joining* p *to* q *and such that* Γ *and* $\Gamma(\alpha)$ *in* p *form the angle* α. *The segments* $\Gamma(\alpha)$ *depend continuously on* α.

REMARK. Actually, the lemma does hold for any closed geodesic G with the property that half of the length of G, $L(G)/2$, is equal to the minimum of the distance $d(r, C(r))$ between a point r and its cut locus $C(r)$ on M. In this form, the lemma is the essential step in the proof of Theorem 1 of [4]. See also Lemma 1 in [5].

1.4. The lemma can also be used to give a second characterization of the complex projective space $P(\mathbf{C})$ with the usual metric among the Kaehlerian manifolds with positive holomorphic curvature:

THEOREM 3. *Let* M *be a compact Kaehlerian manifold satisfying* (1). *Assume that there is a point* p *on* M *with the following property*: *For each holomorphic* 2*-plane* σ *in the tangent space* M_p *there exists a closed geodesic of length* 2π *which is tangent to* σ *in* p. *Then* M *is the complex projective space with the usual Kaehler metric of constant holomorphic curvature* 1.

REMARK. If M has real dimension 2, i.e., if M is homeomorphic to the 2-sphere, then we assume the existence of just one closed geodesic of length 2π. In this case, the theorem is contained in Theorem 1 of the paper [5].

2. **Proof of Theorem** 1. In this paragraph we consider a compact Kaehlerian manifold M satisfying (1).

2.1. M is simply connected. This follows from an argument due to Synge in the case of a Riemannian manifold. A proof of this fact is contained in the paper of Tsukamoto [7]: Let G be a closed geodesic. Then there are curves H in the neighborhood of G which are shorter than G, and therefore, the fundamental group of M can have trivial elements only. One gets the neighboring curves H by considering the 2-dimensional strip formed by short geodesic segments orthogonal to G and tangent to the holomorphic plane which is tangent to G. Since the holomorphic plane is invariant under parallel translation along G and since hol $K>0$, it follows that a parallel curve to G on this strip is shorter than G.

2.2. Assume (1) and assume that there is a closed geodesic G on M of length $L(G)<2\pi$. Moreover, let G have minimal length among the closed geodesics on M. As has been shown in [4], $L(G)/2$ then is the minimum distance between a point r and its cut locus $C(r)$. From the argument in §2.1 we have the existence of a family of curves $H\neq G$ neighboring G, each of which is shorter than G. This is sufficient to conclude, as has been carried out in [4], that two points

p and q on G which are opposite to each other, are conjugate to each other with respect to each of the two parts of G running from p to q. But this yields a contradiction, since these parts have length $L(G)/2 < \pi$, whereas the curvature K is restricted by $K \leqq 1$. This proves (ii).

2.3. The assumption that there is a point p on M such that the distance $d(p, C(p))$ between p and its cut locus $C(p)$ is $< \pi$ implies the existence of a closed geodesic of length $< 2\pi$ (cf. [**4**]) and therefore a contradiction to (ii). This proves (iii) and, at the same time, (iv).

2.4. The Theorem (v) of Tsukamoto is an immediate consequence of the fact that, on a geodesic ray G starting at a point p, the first conjugate point of p must come at a distance $\leqq \pi/H^{1/2}$, if $0 < H \leqq \operatorname{hol} K$.

3. **Proof of Theorem** 2. In this paragraph we assume M to be a compact Kaehlerian manifold satisfying (1) and with diameter $d(M) = \pi$.

3.1. Let p be a point of M and let Γ be a geodesic segment of length π emanating from p. Γ meets the cut locus $C(p)$ in its endpoint q, since, on the one hand, according to Theorem 1, no point of Γ with distance $< \pi$ from p can belong to $C(p)$ and since, on the other hand, $d(M) = \pi$ and therefore no geodesic segment of length $> \pi$ can be a curve of minimal length joining its endpoints.

We will prove that the curvature of the holomorphic 2-planes tangent to Γ is constant $= 1$. Since p and Γ have been chosen arbitrarily, we will then have $\operatorname{hol} K = 1$, and therefore $M = P(\mathbf{C})$.

3.2. Proposition 1 (Berger [**1**]). *Let M be a compact Riemannian manifold. Let p and q be two points on M with maximal distance $d(p, q) = d(M)$. For each unit vector A in the tangent space M_p at p there exists a geodesic segment* E *of length $d(p, q)$ joining p to q such that the initial direction of* E *in p forms with A an angle $\leqq \pi/2$.*

Proof. Consider a sequence of points $p_i \neq p$ on the geodesic starting from p in the direction A and such that $\lim p_i = p$. For each p_i take a segment of length $d(p_i, q)$ joining p_i to q. The set of these segments has a segment of accumulation, E. E is a segment of length $d(p, q)$ joining p to q, and since p and q have maximal distance on M we have from the construction of E that the initial direction of E must form with A an angle $\leqq \pi/2$.

3.3. We return to the proof of Theorem 2. From Proposition 1 we have that, besides Γ, there exists a second segment Λ of length π joining p to q. We now claim (cf. Berger [**1**]):

Proposition 2. *Let M be a compact, orientable Riemannian mani-*

fold of even dimension and assume for the Riemannian curvature K the relations $0<K\leqq 1$. Let p and q be two points of M which are joined by two different geodesic segments Γ and Δ of length π. Then for one of the two segments, let us say for Δ, the following does hold: There is a 2-plane σ in M_p which contains the initial directions of Γ and Λ in M_p such that, for all 2-planes along Δ which one gets by parallel translating σ along Δ, the Riemannian curvature is equal to 1.

COROLLARY. *Let M be a compact Kaehlerian manifold satisfying* (1). *Let p and q be two points of M which are joined by two different geodesic segments Γ and Δ of length π. Then, for one of these segments, let us say for Δ, the following does hold: The holomorphic 2-planes tangent to Δ have constant curvature equal to* 1. *At the point p this holomorphic 2-plane contains both, the initial direction of Γ and of Δ.*

PROOF. (i) We first consider the case that the initial directions A and B of Γ and Δ in M_p form an angle $<\pi$. Let σ be the 2-plane spanned by A and B. From Theorem 1 we have that no point r which has distance $<\pi$ from q can be joined to q by two different geodesic segments of minimal length. Therefore, there does not exist a point $r\neq p$, close to p on a geodesic starting from p in a direction which forms the same angle $<\pi/2$ with both A and B such that r can be joined to q by two geodesic segments Γ' and Δ', where Γ' is neighboring Γ and Δ' is neighboring Δ. That is to say, p must be conjugate to q with respect to one of the segments Γ and Δ, let us say, with respect to Δ. More precisely, since $K\leqq 1$ and since Δ has the length π, there does exist a solution $u\neq 0$ of the Jacobi equations along Δ which vanishes at q and vanishes again at p and arrives at p in a direction which is contained in the 2-plane σ spanned by A and B. The field of 2-planes along Δ, spanned by the Jacobi vector u and by the tangent vector to Δ, is invariant under parallel translation and each 2-plane of the field has constant Riemannian curvature equal to 1.

(ii) Next consider the case that Γ and Δ form an angle π at p and an angle $<\pi$ at q. By interchanging p and q in the argument used in (i) we get the conclusion of the proposition. The remaining alternative is that the two geodesic segments Γ and Δ together form a closed geodesic of length 2π. Onto this situation, we can apply the lemma and again get that q is conjugate to p with respect to Γ as well as with respect to Δ. The remainder of the argument goes as in (i).

(iii) For the proof of the corollary we remark that, according to Berger [2], the assumption (1) implies $K=K(\sigma)\leqq 1$ and $K(\sigma)=1$ for holomorphic 2-planes σ only. Therefore, if we have a field of tangent 2-

planes along Δ with Riemannian curvature equal to 1, these 2-planes must be holomorphic ones. The existence of such a field of 2-planes, however, is proved by the same argument that we have used in (i) and (ii); especially, we have for Kaehlerian manifolds satisfying (1) again the conclusions of the lemma as has been pointed out in §2.2.

3.4. We now proceed with the proof of Theorem 2. From the complex structure of M we have an orientation in the holomorphic 2-planes, i.e., we can speak of an oriented angle in such a 2-plane.

According to the corollary of Proposition 2, the initial directions of Γ and Δ at p as well as at q are in a holomorphic 2-plane. We may assume that the initial directions of Γ and Δ at p form an oriented angle $\gamma \leqq \pi$. By replacing if necessary the segment Δ by a segment Δ' according to Proposition 1 we will assume furthermore the oriented angle from the initial direction of Δ in q to the initial direction of Γ in q is also $\leqq \pi$. We claim that under these circumstances the conclusions of the lemma do hold: *For each* α, $0 \leqq \alpha \leqq \gamma$, *there exists a geodesic segment* $\Gamma(\alpha)$ *of length* π *joining* p *to* q *and such that* Γ *and* $\Gamma(\alpha)$ *in* p *form the angle* α. From the corollary to Proposition 2 we then have: *The initial direction of* $\Gamma(\alpha)$ *in* p *is in the holomorphic* 2-*plane spanned by the initial directions of* Γ *and* Δ *in* p.

The proof of this version of the lemma for Kaehlerian manifolds satisfying (1) goes along the same lines as the proof given in §2.2: Take the holomorphic 2-strip tangent to the closed curve $G = \Gamma \cup \Delta$ of length 2π. On the "inner" side of this strip consider the parallel curves $H \neq G$. Each of these curves H is shorter than 2π; at this point we use the fact that Γ and Δ form a two-angle with oriented angles $\leqq \pi$. The curves H have G as a limit curve and from the argument given in [4, pp. 661–662], we have the existence of the curves $\Gamma(\alpha)$, $0 \leqq \alpha \leqq \gamma$.

Consequently, q is conjugate to p with respect to Γ as well as to Δ. But then we have, as in §3.3, the existence of a field of tangent 2-planes along Γ and Δ with Riemannian curvature equal to 1. According to Berger [2], these planes must be holomorphic ones. Hence we have proved the statement at the end of §3.1 and therefore, Theorem 2.

4. Proof of Theorem 3. In this paragraph we make the assumptions stated in Theorem 3.

4.1. Let σ be a holomorphic 2-plane in M_p and let G be a closed geodesic of length 2π tangent to the plane σ. We take G as being formed by two geodesic segments Γ and Δ of length π which join p to the point q opposite to p on G.

Applying the lemma (more precisely; the version of the lemma for Kaehlerian manifolds satisfying (1)) onto this situation we get a family of segments $\Gamma(\alpha)$, $0 \leqq \alpha \leqq \pi$, each of which has length π and joins p to q. From the corollary of Proposition 2 we have that the initial directions of these segments $\Gamma(\alpha)$ are in the holomorphic 2-plane σ. In the same way as the proof of Satz 2.3 in [5], this statement can be sharpened in the following way: For each value α, $0 < \alpha < \pi$, there are precisely two segments, $\Gamma(\alpha)$ and $\Gamma'(\alpha)$, of length π joining p to q which form with Γ in p the nonoriented angle α, i.e., one of each "side" of Γ.

But this means that all geodesic segments of length π which start from p in a direction contained in the holomorphic 2-plane σ meet again at the opposite point q of p on G, thus forming a totally geodesic 2-sphere of constant curvature equal to 1. This holds for each holomorphic 2-plane in M_p, and therefore, M is the complex projective space $P(\boldsymbol{C})$.

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