ON RELATIVELY NONATOMIC MEASURES

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1. Introduction and Summary. Let $\Omega$ be a set, $\mathcal{A}$ a $\sigma$-algebra of subsets of $\Omega$, and $m$ be a measure defined on $\mathcal{A}$ whose range $R$ is a subset of $k$-dimensional Euclidean space $E_k$, i.e., $m$ is a completely additive set function defined on $\mathcal{A}$ such that each component of $m$ is nonnegative. If each component of $m$ is nonatomic it was shown by Lyapunov [3], Halmos [2], and Blackwell [1] that $R$ is a convex set. Now let $\mathcal{A}$ be the class of Borel subsets of the real line, let $m = (m_1, m_2)$ where $m_1$ is Lebesgue measure and $m_2(A)$ counts the number of integers in the set $A$. Then $R$ is not convex, in fact $R$ is the set of points $\{(x, y)\}$ where $x \geq 0$ and $y$ is a nonnegative integer. Yet if $(x, y) \in R$ and $(x, y)$ lies on the line segment connecting the zero vector and $m(A)$ it is easily seen that there exists $A' \subset A$ such that $m(A') = (x, y)$. In this note we give a definition of relative nonatomicity which covers situations of this kind and prove the analogue of the convexity theorem.

2. Definitions and Results. We shall assume throughout that every subset of $\Omega$ that is discussed is measurable, i.e., an element of $\mathcal{A}$.

Definition 1. Let $R$ be the range of $m$. $m$ is nonatomic relative to $R$ if for every set $A$ and every number $a$ with $0 < a < 1$ such that $am(A) \in R$ there exists $A' \subset A$ and a number $a'$ with $0 < a' < 1$ such that $m(A') = a'm(A)$.

Definition 2. Let $A$ be a set. We define $R(A) = \{r \in R | r = am(A) \text{ with } 0 < a \leq 1\}$. Let $a_0 = \inf \{a | a > 0, \, am(A) \in R\}$. Then we define $r_0(A) = a_0m(A)$.

Theorem. Let $m$ be nonatomic relative to $R$. Let $A$ be a set and $r \in R(A)$. Then there exists $B \subset A$ with $r = m(B)$.

The proof of the theorem will proceed by way of several lemmas.

Lemma 1. In Definition 1 we may choose $A'$ and $a'$ such that $a' \leq a$.

Proof. Suppose the conclusion of the lemma is false. Let $A' \subset A$ such that $m(A') = a'm(A)$ with $0 < a' < 1$. Then $m(A - A') = (1 - a')m(A)$ and $a' > a, 1 - a' > a$. Hence $a < 1/2$. Now $r = am(A) = (\alpha/\alpha')a'm(A') = (\alpha/\alpha')m(A')$, and similarly

$$r = [\alpha/(1 - \alpha')]m(A - A').$$

Hence we may apply the definition and the above procedure separately to $A'$ and $A - A'$. After some manipulation we obtain $\alpha < 1/4$.

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Repeating this process indefinitely yields $\alpha = 0$, which is a contradiction.

**Lemma 2.** Suppose $m$ is nonatomic relative to $R$. Then for every set $A$ there exists $B \subseteq A$ such that $m(B) = r_0(A) = \alpha m(A)$ for some $\alpha$, with $0 \leq \alpha \leq 1$.

**Proof.** If $r_0(A)$ is the null vector choose $B$ to be the empty set and $\alpha = 0$, and if $r_0(A) = m(A)$ choose $B = A$ and $\alpha = 1$. If $r_0(A) \in R(A)$ the conclusion follows from Lemma 1. Otherwise there exists a strictly decreasing sequence of numbers $\{\alpha_n\}$ with $0 < \alpha_n < 1$ such that $r_0(A) = \lim_n \alpha_n m(A)$, and such that $\alpha_m m(A) \in R(A)$. But then it follows again from Lemma 1 that there exists a sequence of number $\{\alpha_n'\}$ and a sequence of sets $\{B_n\}$ which may be chosen so that $A \supseteq B_1 \supseteq B_2 \supseteq \cdots$ such that $\lim_n m(B_n) = \lim_n \alpha_n' m(A) = r_0(A)$. If $B = \lim_n B_n$ then clearly $B$ satisfies the conclusion of the lemma.

**Lemma 3.** Suppose $m$ is nonatomic relative to $R$. Let $A$ be a set and suppose $B \subseteq A$ satisfies the conclusion of Lemma 2, i.e., $m(B) = r_0(A)$. Then either $m(B)$ is the null vector or $m(A)$ is an integral multiple of $m(B)$.

**Proof.** Suppose $m(B)$ is not the null vector, i.e., $m(B) = \alpha m(A)$ for some $\alpha$ with $0 < \alpha \leq 1$. If $\alpha = 1$ we are done. Assume then that $0 < \alpha < 1$. Choose $B' \subseteq A - B$ so as to satisfy Lemma 2, i.e., $m(B') = r_0(A - B) = \alpha' m(A - B) = \alpha'(1 - \alpha) m(A)$ for some $\alpha'$ with $0 \leq \alpha' \leq 1$. From the definition of the function $r_0$ it follows that $\alpha'(1 - \alpha) \geq \alpha$. If $\alpha < \alpha'(1 - \alpha)$ we may write $m(B) = [\alpha/(\alpha'(1 - \alpha))] m(B')$. It then follows from the definition of relative nonatomicity that there exist $B'' \subseteq B'$ and $\alpha''$ with $0 < \alpha'' < 1$ such that $m(B'') = \alpha'' m(B')$. But this contradicts the fact that $B'$ is minimal for $A - B$, i.e., $m(B') = r_0(A - B)$. Hence $m(B') = m(B)$. Now if $m(A - B - B')$ is the null vector we are done. Otherwise we repeat this process. But clearly this must stop in a finite number of steps, and the lemma is proved. By the same techniques we have immediately

**Lemma 4.** Suppose $m$ is nonatomic relative to $R$. Let $A$ be a set and suppose $r_0(A)$ is not the null vector. Then $A$ is the union of finitely many disjoint sets, each having measure $r_0(A)$.

**Lemma 5.** Suppose $m$ is nonatomic relative to $R$. Let $A$ be a set and suppose $r_0(A)$ is not the null vector. Then every $r \in R(A)$ is a positive integral multiple of $r_0(A)$.

**Proof.** Let $r \in R(A)$, i.e., $r = \alpha m(A)$ with $0 < \alpha \leq 1$. From Lemma 3
it follows that there exists $B \subseteq A$ such that $m(A) = nm(B) = nr_0(A)$ for some positive integer $n$. If $\alpha = 1$ there is nothing to prove. Assume then that $0 < \alpha < 1$ and that $\alpha = (k+c)/n$ where $k$ is an integer with $1 \leq k < n$ and $c$ is a number with $0 < c < 1$. The case $k = 0$ is impossible for in that case $\alpha < 1/n$. Now $r \in R$ and hence there exists a set $C$ such that $r = m(C)$. Now consider $r_0(C) = \alpha' m(C) = \alpha' \alpha m(A)$ for some $\alpha'$ with $0 \leq \alpha' \leq 1$. If $\alpha' = 0$ then $r_0(A)$ is the null vector which is contrary to the hypothesis. Consequently $\alpha' > 0$ and $m(C) = \text{im}(C_i)$ for some set $C_i \subseteq C$ and some positive integer $i$, from Lemma 3. Now $m(B) = (1/n)m(A) = (1/(k+c))m(C)$ and hence $(1/i) \leq (1/(k+c))$. Since $k+c$ is not an integer we have $k+c < i$. But then $m(C_i) = [(k+c)/(in)]m(A)$ and $(k+c)/(in) < 1/n$ which contradicts the minimality of $1/n$. Hence $k+c$ must be an integer and the lemma is proved.

**Proof of the theorem.** If $r_0(A)$ is not the null vector the conclusion of the theorem follows at once from Lemma 4 and Lemma 5. Suppose then that $r_0(A)$ is the null vector. Now $r = \alpha m(A)$ for some $\alpha$ with $0 \leq \alpha \leq 1$. If $\alpha = 0$ or $\alpha = 1$ the conclusion is trivial. Assume then that $0 < \alpha < 1$. Let $\mathcal{F} = \{B \subseteq A \mid m(B) = \beta m(A) \text{ with } 0 < \beta \leq \alpha\}$. We partially order $\mathcal{F}$ by saying that $B_1 < B_2$ if $B_1 \subseteq B_2$ and if the corresponding $\beta_1$ and $\beta_2$ satisfy $\beta_1 < \beta_2$. If $\mathcal{F}'$ is a linearly ordered subfamily of $\mathcal{F}$ it is easily seen that $\mathcal{F}'$ has an upper bound in $\mathcal{F}$. Consequently Zorn's lemma applies. Let $B$ be a maximal element of $\mathcal{F}$ and suppose $m(B) = \beta m(A)$. We shall show that $\beta = \alpha$. Suppose $\beta < \alpha$. Since $r_0(A)$ is the null vector it follows easily that $r_0(A - B)$ is the null vector. Hence we can find an arbitrarily small positive number $\gamma$ and a corresponding set $B' \subseteq A - B$ such that $m(B') = \gamma m(A - B) = \gamma(1-\beta)m(A)$. Let $B'' = B \cup B'$. Then $m(B'') = [\beta + \gamma(1-\beta)]m(A)$ and by choosing $\gamma$ sufficiently small we violate the maximality of $B$. Thus $\beta = \alpha$ and theorem is proved.

**References**

