ON RELATIVELY NONATOMIC MEASURES

J. R. BLUM

1. Introduction and summary. Let \( \Omega \) be a set, \( \mathcal{A} \) a \( \sigma \)-algebra of subsets of \( \Omega \), and \( m \) be a measure defined on \( \mathcal{A} \) whose range \( R \) is a subset of \( k \)-dimensional Euclidean space \( E_k \), i.e., \( m \) is a completely additive set function defined on \( \mathcal{A} \) such that each component of \( m \) is nonnegative. If each component of \( m \) is nonatomic it was shown by Lyapunov [3], Halmos [2], and Blackwell [1] that \( R \) is a convex set. Now let \( \mathcal{A} \) be the class of Borel subsets of the real line, let \( m = (m_1, m_2) \) where \( m_1 \) is Lebesgue measure and \( m_2(A) \) counts the number of integers in the set \( A \). Then \( R \) is not convex, in fact \( R \) is the set of points \( \{ (x, y) \} \) where \( x \geq 0 \) and \( y \) is a nonnegative integer. Yet if \( (x, y) \in R \) and \( (x, y) \) lies on the line segment connecting the zero vector and \( m(A) \) it is easily seen that there exists \( A' \subseteq A \) such that \( m(A') = (x, y) \). In this note we give a definition of relative nonatomicity which covers situations of this kind and prove the analogue of the convexity theorem.

2. Definitions and results. We shall assume throughout that every subset of \( \Omega \) that is discussed is measurable, i.e., an element of \( \mathcal{A} \).

**Definition 1.** Let \( R \) be the range of \( m \). \( m \) is nonatomic relative to \( R \) if for every set \( A \) and every number \( a \) with \( 0 < a < 1 \) such that \( am(A) \in R \) there exists \( A' \subseteq A \) and a number \( a' \) with \( 0 < a' < 1 \) such that \( m(A') = a'm(A) \).

**Definition 2.** Let \( A \) be a set. We define \( R(A) = \{ r \in R \mid r = am(A) \} \) with \( 0 < a \leq 1 \). Let \( \alpha_0 = \inf \{ a \mid a > 0, \ am(A) \in R \} \). Then we define \( r_0(A) = \alpha_0 m(A) \).

**Theorem.** Let \( m \) be nonatomic relative to \( R \). Let \( A \) be a set and \( r \in R(A) \). Then there exists \( B \subseteq A \) with \( r = m(B) \).

The proof of the theorem will proceed by way of several lemmas.

**Lemma 1.** In **Definition 1** we may choose \( A' \) and \( \alpha' \) such that \( \alpha' \leq \alpha \).

**Proof.** Suppose the conclusion of the lemma is false. Let \( A' \subseteq A \) such that \( m(A') = \alpha'm(A) \) with \( 0 < \alpha' < 1 \). Then \( m(A - A') = (1 - \alpha')m(A) \) and \( \alpha' > \alpha \), \( 1 - \alpha' > \alpha \). Hence \( \alpha < 1/2 \). Now \( r = \alpha m(A) = (\alpha/\alpha')\alpha'm(A) = (\alpha/\alpha')m(A') \), and similarly

\[ r = [\alpha/(1 - \alpha')]m(A - A') \text{.} \]

Hence we may apply the definition and the above procedure separately to \( A' \) and \( A - A' \). After some manipulation we obtain \( \alpha < 1/4 \).

Received by the editors March 2, 1960 and, in revised form, May 27, 1960.

457
Repeating this process indefinitely yields \( a = 0 \), which is a contradiction.

**Lemma 2.** Suppose \( m \) is nonatomic relative to \( R \). Then for every set \( A \) there exists \( B \subset A \) such that \( m(B) = r_0(A) = \alpha m(A) \) for some \( \alpha \), with \( 0 \leq \alpha \leq 1 \).

**Proof.** If \( r_0(A) \) is the null vector choose \( B \) to be the empty set and \( \alpha = 0 \), and if \( r_0(A) = m(A) \) choose \( B = A \) and \( \alpha = 1 \). If \( r_0(A) \in R(A) \) the conclusion follows from Lemma 1. Otherwise there exists a strictly decreasing sequence of numbers \( \{ \alpha_n \} \) with \( 0 < \alpha_n < 1 \) such that \( r_0(A) = \lim_n \alpha_n m(A) \), and such that \( \alpha_n m(A) \in R(A) \). But then it follows again from Lemma 1 that there exists a sequence of numbers \( \{ \alpha'_n \} \) with \( 0 < \alpha'_n < 1 \) such that \( r_0(A) = \lim_n \alpha'_n m(A) = r_0(A) \). If \( B = \lim_n B_n \) then clearly \( B \) satisfies the conclusion of the lemma.

**Lemma 3.** Suppose \( m \) is nonatomic relative to \( R \). Let \( A \) be a set and suppose \( B \subset A \) satisfies the conclusion of Lemma 2, i.e., \( m(B) = r_0(A) \). Then either \( m(B) \) is the null vector or \( m(A) \) is an integral multiple of \( m(B) \).

**Proof.** Suppose \( m(B) \) is not the null vector, i.e., \( m(B) = \alpha m(A) \) for some \( \alpha \) with \( 0 < \alpha \leq 1 \). If \( \alpha = 1 \) we are done. Assume then that \( 0 < \alpha < 1 \). Choose \( B' \subset A - B \) so as to satisfy Lemma 2, i.e., \( m(B') = r_0(A-B) = \alpha' m(A-B) = \alpha'(1-\alpha) m(A) \) for some \( \alpha' \) with \( 0 \leq \alpha' \leq 1 \). From the definition of the function \( r_0 \) it follows that \( \alpha'(1-\alpha) \geq \alpha \). If \( \alpha < \alpha'(1-\alpha) \) we may write \( m(B) = \lceil \alpha / (\alpha'(1-\alpha)) \rceil m(B') \). It then follows from the definition of relative nonatomicity that there exist \( B'' \subset B' \) and \( \alpha'' \) with \( 0 < \alpha'' < 1 \) such that \( m(B'') = \alpha'' m(B') \). But this contradicts the fact that \( B' \) is minimal for \( A - B \), i.e., \( m(B') = r_0(A-B) \). Hence \( m(B') = m(B) \). Now if \( m(A-B-B') \) is the null vector we are done. Otherwise we repeat this process. But clearly this must stop in a finite number of steps, and the lemma is proved. By the same techniques we have immediately

**Lemma 4.** Suppose \( m \) is nonatomic relative to \( R \). Let \( A \) be a set and suppose \( r_0(A) \) is not the null vector. Then \( A \) is the union of finitely many disjoint sets, each having measure \( r_0(A) \).

**Lemma 5.** Suppose \( m \) is nonatomic relative to \( R \). Let \( A \) be a set and suppose \( r_0(A) \) is not the null vector. Then every \( r \in R(A) \) is a positive integral multiple of \( r_0(A) \).

**Proof.** Let \( r \in R(A) \), i.e., \( r = \alpha m(A) \) with \( 0 < \alpha \leq 1 \). From Lemma 3
it follows that there exists $B \subset A$ such that $m(A) = nm(B) = nr_0(A)$ for some positive integer $n$. If $\alpha = 1$ there is nothing to prove. Assume then that $0 < \alpha < 1$ and that $\alpha = (k+c)/n$ where $k$ is an integer with $1 \leq k < n$ and $c$ is a number with $0 < c < 1$. The case $k = 0$ is impossible for in that case $\alpha < 1/n$. Now $r \in \mathbb{R}$ and hence there exists a set $C$ such that $r = m(C)$. Now consider $r_0(C) = \alpha'm(C) = \alpha'\alpha m(A)$ for some $\alpha'$ with $0 \leq \alpha' \leq 1$. If $\alpha' = 0$ then $r_0(A)$ is the null vector which is contrary to the hypothesis. Consequently $\alpha' > 0$ and $m(C) = im(C_i)$ for some set $C_i \subset C$ and some positive integer $i$, from Lemma 3. Now $m(B) = (1/n)m(A) = (1/(k+c))m(C)$ and hence $(1/i) \leq (1/(k+c))$. Since $k+c$ is not an integer we have $k+c < i$. But then $m(C_i) = (k+c)/(in)m(A)$ and $(k+c)/(in) < 1/n$ which contradicts the minimality of $1/n$. Hence $k+c$ must be an integer and the lemma is proved.

Proof of the theorem. If $r_0(A)$ is not the null vector the conclusion of the theorem follows at once from Lemma 4 and Lemma 5. Suppose then that $r_0(A)$ is the null vector. Now $r = \alpha m(A)$ for some $\alpha$ with $0 \leq \alpha \leq 1$. If $\alpha = 0$ or $\alpha = 1$ the conclusion is trivial. Assume then that $0 < \alpha < 1$. Let $\mathcal{F} = \{B \subset A \mid m(B) = \beta m(A) \text{ with } 0 < \beta \leq \alpha\}$. We partially order $\mathcal{F}$ by saying that $B_1 < B_2$ if $B_1 \subset B_2$ and if the corresponding $\beta_1$ and $\beta_2$ satisfy $\beta_1 < \beta_2$. If $\mathcal{F}'$ is a linearly ordered subfamily of $\mathcal{F}$ it is easily seen that $\mathcal{F}'$ has an upper bound in $\mathcal{F}$. Consequently Zorn's lemma applies. Let $B$ be a maximal element of $\mathcal{F}$ and suppose $m(B) = \beta m(A)$. We shall show that $\beta = \alpha$. Suppose $\beta < \alpha$. Since $r_0(A)$ is the null vector it follows easily that $r_0(A - B)$ is the null vector. Hence we can find an arbitrarily small positive number $\gamma$ and a corresponding set $B' \subset A - B$ such that $m(B') = \gamma m(A - B) = \gamma (1 - \beta)m(A)$. Let $B'' = B \cup B'$. Then $m(B'') = [\beta + \gamma (1 - \beta)]m(A)$ and by choosing $\gamma$ sufficiently small we violate the maximality of $B$. Thus $\beta = \alpha$ and theorem is proved.

References


Sandia Corporation