Case (5) and (6) continuous spectrum $0 < \lambda < \infty$, point spectrum
$-\infty < \lambda < 0$,
Case (7) and (8) point spectrum $0 < \lambda < \infty$, continuous spectrum
$-\infty < \lambda < 0$,
Case (9) point spectrum $Q_1 < \lambda < Q_2$, continuous spectrum
$\lambda < \min \{Q_1, Q_2\}$, $\lambda > \max \{Q_1, Q_2\}$.

References


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**A METHOD OF APPROXIMATING THE ZEROS OF FUNCTIONS BY QUADRATIC FORMULAS**

**STEPHEN KULIK**

1. **Introduction.** The problem of approximating two zeros of a given function by solving a quadratic equation was discussed in a number of papers [1; 2; 4; 5; 7]. In this paper we present a general method of deriving quadratic equations the two roots of which would approximate two zeros of an analytic function $f(z)$. A function $f(z)/g(z, u)$, instead of $f(z)$, is considered, where $g(z, u)$ is another appropriately chosen analytic function. By varying the parameter $u$, and keeping the initial approximation to the zeros unchanged, the final approximations can be improved or another pair of zeros approximated. The exact values of the zeros are determined as limits of the expressions approximating them.

2. **The general method.** Let $f(z)$ and $g(z, u)$ be analytic functions
within and on the circle \( C \), where \( f(z) \) has zeros, simple or multiple. The function \( g(z, u) \) may have zeros in common with \( f(z) \). However, we assume that those zeros of \( f(z) \) which we desire to calculate are simple zeros of \( f(z)/g(z, u) \).

Let the expansion of \( g(z, u)/f(z) \) into partial fractions be

\[
g/f = \sum A_j^{(1)}/h_j^{m_j} + \psi_1,
\]

where \( g = g(z, u) \), \( f = f(z) \), \( h_j = z - a_j \), \( A_j \) is a polynomial in \( z - a_j \) of a degree not higher than \( m_j - 1 \), \( m_j \) being the multiplicity of the zero \( a_j \); \( \psi_1 = \psi_1(z) \) is analytic within and on \( C \); and the summation is carried out over all \( a_j \).

On differentiating (1) \((n-1)\) times and multiplying it by \((-1)^{n-1}/(n-1)!\), we obtain

\[
H_n = \sum A_j^{(n)}/h_j^{n+m_j-1} + \psi_n,
\]

where \( A_j^{(n)} \) again is a polynomial in \( z - a_j \) of a degree not higher than \( m_j - 1 \), \( \psi_n = (-1)^{n-1}\psi_1^{(n-1)}/(n-1)! \), and \( H_n = Q_n/f^n \). The function \( Q_n \) is the determinant

\[
Q_n = \begin{vmatrix}
g & f & 0 & \cdots \\
g' & f' & f & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
g^{(n-1)}/(n-1)! & f^{(n-1)}/(n-1)! & f^{(n-2)}/(n-2)! & \cdots & f'
\end{vmatrix}
\]

It can be evaluated recursively, [3],

\[
Q_n = f'Q_{n-1} - ff''Q_{n-2}/2! + \cdots + (-f)^{n-2}f^{(n-1)}Q_1/(n-1)! + (-f)^{n-1}g^{(n-1)}Q_0/(n - 1)!
\]

\( Q_0 = 1, \quad Q_1 = g, \quad Q_2 = f'Q_1 - ff''Q_0. \)

We assume that \( a_1 \) and \( a_2 \) are the two simple zeros of \( f(z)/g(z, u) \) which we wish to calculate and rewrite (2) in the form

\[
H_n = (A_1/h_1^n + A_2/h_2^n)(1 + \beta_n),
\]

where

\[
\beta_n = h_1^n h_2^n \left( \sum' A_j^{(n)}/h_j^{n+m_j-1} + \psi_n \right)/(A_1 h_1^n + A_2 h_2^n);
\]

the prime on the summation sign means that the terms for \( j = 1, 2 \), are to be omitted, and \( A_1, A_2 \) stand for \( A_1^{n+m_1-1} \) and \( A_2^{n+m_2-1} \).

Now if \( u \) is a fixed number and \( z \) is given a numerical value such that

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for any \( v \) on \( C \), then

\[
\lim_{n \to \infty} \beta_n = 0 \quad \text{as} \quad n \to \infty.
\]

Using (5) we may write the following equations:

\[
\begin{align*}
A_1/h_1^{n-1} + A_2/h_2^{n-1} &= (1 + \alpha_{n-1})H_{n-1}, \\
A_1/h_1^n + A_2/h_2^n &= (1 + \alpha_n)H_n, \\
A_1/h_1^{n+1} + A_2/h_2^{n+1} &= (1 + \alpha_{n+1})H_{n+1},
\end{align*}
\]

and, eliminating \( A_1/h_1^{n-1} \) and \( A_2/h_2^{n-2} \) between them, obtain

\[
(1 + \alpha_{n-1})H_{n-1} - (1 + \alpha_n)(h_1 + h_2)H_n + (1 + \alpha_{n+1})h_1h_2H_{n+1} = 0,
\]

where

\[
\lim_{n \to \infty} \alpha_n = 0 \quad \text{as} \quad n \to \infty.
\]

Equation (10) shows that \( h_1 \) and \( h_2 \) satisfy a quadratic equation. Therefore, we may write

\[
\begin{align*}
h^2 + ph + q &= 0, \\
H_{n-1}^1 + pH_n^1 + qH_{n+1}^1 &= 0, \\
H_n^1 + pH_{n+1}^1 + qH_{n+2}^1 &= 0,
\end{align*}
\]

where

\[
p = -(h_1 + h_2), \quad q = h_1h_2, \quad \text{and} \quad H_j^1 = (1 + \alpha_j)H_j, \quad j = n - 1, n, n + 1, n + 2.
\]

Equating the determinant of (12) to zero, we get the following equation which is satisfied by \( h_1 \) and \( h_2 \):

\[
\begin{vmatrix}
h^2 & h & 1 \\
H_{n-1}^1 & H_n^1 & H_{n+1}^1 \\
H_n^1 & H_{n+1}^1 & H_{n+2}^1
\end{vmatrix} = 0.
\]

The equation approximating \( h_1 \) and \( h_2 \) may now be written,

\[
\begin{vmatrix}
h^2 & h & 1 \\
H_{n-1} & H_n & H_{n+1} \\
H_n & H_{n+1} & H_{n+2}
\end{vmatrix} = 0,
\]
or

\[
\begin{vmatrix}
    h^2 & fh & f^2 \\
    Q_{n-1} & Q_n & Q_{n+1} \\
    Q_n & Q_{n+1} & Q_{n+2}
\end{vmatrix} = 0.
\]

Note that we have not introduced a separate symbol for the approximating value of \( h \).

From (14) also follows

\[
(17) \quad a_1, a_2 = z - f \lim_{n \to \infty} B_n = (B_n - 4A_nA_{n-1})^{1/2}/2A_n,
\]

where

\[
(18) \quad A_n = Q_nQ_{n+2} - Q_{n+1}^2, \quad B_n = Q_{n-1}Q_{n+2} - Q_nQ_{n+1}.
\]

The quadratic equation (16) is satisfied exactly by the zeros of a quadratic polynomial in \( z \) if \( g(z, u) \) is a polynomial in \( z \). If the degree of \( g(z, u) \) is \( k \), it is satisfied starting with the lowest \( Q_{n-1} = Q_k \). This follows from the fact that (16) would coincide with (14) starting with \( n - 1 = k \).

We consider in more detail (16) when \( g(z, u) = 1 \). In this case the determinant (3) is reduced to a determinant of order \( n - 1 \). We denote it by \( P_{n-1} \) and write the recursive relation between \( P_n \) and the lower determinants.

\[
(19) \quad P_n = f'P_{n-1} - f''P_{n-2}/2! + \cdots + (-1)^{n-1}f^{(n)}P_0/n!, \quad P_0 = 1.
\]

The equation corresponding to (16), with \( n \) increased by one, we rewrite in the form

\[
(20) \quad \begin{vmatrix} P_{n+1} & P_n \\ P_{n+2} & P_{n+1} \end{vmatrix} h^2 \quad \begin{vmatrix} P_n & P_{n-1} \\ P_{n+2} & P_{n+1} \end{vmatrix} fh + \begin{vmatrix} P_n & P_{n-1} \\ P_{n+1} & P_n \end{vmatrix} f^2 = 0.
\]

There is no difficulty in showing that

\[
(21) \quad \begin{vmatrix} P_n & P_{n-1} \\ P_{n+1} & P_n \end{vmatrix} = f^nK_n
\]

and

\[
(22) \quad \begin{vmatrix} P_n & P_{n-1} \\ P_{n+2} & P_{n+1} \end{vmatrix} = f^nL_{n+1}, \quad L_{n+1} = f'K_n - fM_n,
\]

where
With these notations (20) can now be presented in a shorter form,

(26) \[ K_{n+1} h^2 - L_{n+1} h + f K_n = 0, \]

(27) \[ K_{n+1} h^2 + (f M_n - f' K_n) h + f K_n = 0. \]

Further simplifications of (26) or (27) are possible in some particular cases. As an illustration, we take the cubic trinomial \( z^3 + pz + q \).

For \( s = 0, f = q, f' = p, f''/2! = 0, f'''/3! = 1, f^{(n)} = 0 \), \( n = 4, 5, \cdots \), and it may be seen at once that

(28) \[ K_n = -pK_{n-2} + qK_{n-3} \]

and

(29) \[ L_{n+1} = -K_{n+2}. \]

Thus, the two zeros of \( z^3 + pz + q \) which are smaller in absolute value can be approximated by solving

(30) \[ K_{n+1} h^2 + K_{n+2} h + q K_n = 0. \]

The function \( K_n \) can be evaluated recursively by using (28) or by solving (28) for \( K_n \) in terms of \( p \) and \( q \) (see [6]). The solution for \( K_n \) is:

(31) \[ K_n = (-p)^r \left[ 1 + \sum_{k=1}^{m} (-1)^k \left( \frac{r - k}{2k} \right) X^k \right] \quad \text{for } n \text{ even;} \]

(32) \[ r = \frac{n-1}{2}, \quad m = \left[ \frac{n-2}{6} \right], \quad X = q^2/p^3. \]
The first few values of $K_n$ are

$$
K_0 = 1, \quad K_1 = 0, \quad K_2 = -p, \quad K_3 = q, \quad K_4 = p^2, \quad K_5 = -2pq,
$$

$$
K_6 = -p^3 + q^2, \quad K_7 = 3p^2q, \quad K_8 = p^4 - 3pq^2, \quad K_9 = -4p^3q + q^3,
$$

$$
K_{10} = -p^5 + 6pq^2, \quad K_{11} = 5p^4q - 4pq^4, \quad K_{12} = p^6 - 10p^3q^2 + q^4.
$$

We will be using another simple case in which (16) is satisfied exactly by the zeros of a quadratic polynomial, namely, $g(z, u) = f'(z)$. Using the notation $D_n$ for $Q_n$ the recursive formula is

$$
D_n = f'D_{n-1} - ff''D_{n-2}/2! + \cdots + (-f)^{n-2}f^{(n-1)}D_1/(n - 1)!
$$

(33) \quad + (-f)^{n-1}f^{(n)}D_0/(n - 1)!, \quad n = 3, 4, \cdots ,

$$
D_0 = 1, \quad D_1 = f', \quad D_2 = f'' - ff''.
$$

3. The use of the parameter. The usefulness of introducing a parameter into quadratic formulas will now be illustrated.

Let

$$
q(z, u) = (u - z)^k f'(z),
$$

where $k$ is a positive integer and $u$ an arbitrary number not equal to $z$, $a_1$, or $a_2$. The zeros $a_1$ and $a_2$ may be simple or multiple. By applying (5) we get

$$
H_{n,k} = (m_1e_1/h_1^n + m_2e_2/h_2^n)(1 + \beta_{n,k}),
$$

(35) \quad where $h_1 = z - a_1$, $h_2 = z - a_2$, as before; $e_1 = u - a_1$, $e_2 = u - a_2$; $m_1$ and $m_2$ are the multiplicities of $a_1$ and $a_2$ respectively; $H_{n,k} = H_{n,k}(z, u)$, $\beta_{n,k} = \beta_{n,k}(z, u)$.

Now if we assume that $n - k$ = constant and $z$ and $u$ are given such numerical values that

$$
|e_1/h_1| \geq |e_2/h_2| > |e_j/h_j|, \quad j = 3, 4, \cdots ,
$$

(36) \quad and

$$
|e_1/h_1| \geq |e_2/h_2| > |(u - v)/(z - v)|,
$$

for any $v$ on $C$, then $lim \beta_{n,k} = 0$ as $n \rightarrow \infty$.

Let $n - k = 0$; then we can write the approximate equations

$$
m_1(e_1/h_1)^{n-1} + m_2(e_2/h_2)^{n-1} = H_{n-1,n-1},
$$

(37) \quad $m_1(e_1/h_1)^n + m_2(e_2/h_2)^n = H_{n,n},$

$$
m_1(e_1/h_1)^{n+1} + m_2(e_2/h_2)^{n+1} = H_{n+1,n+1}$
and, eliminating \( m_1(e_1/h_1)^{n-1} \) and \( m_2(e_2/h_2)^{n-1} \), we obtain

\[
(38) \quad H_{n-1,n-1} + pH_{n,n} + qH_{n+1,n+1} = 0,
\]

where \( p = -(h_1/e_1 + h_2/e_2) \), \( q = h_1h_2/e_1e_2 \).

We write now the equations

\[
(39) \quad H_{n-1,n-1} + pH_{n,n} + qH_{n+1,n+1} = 0,
\]

\[
H_{n,n} + pH_{n+1,n+1} + qH_{n+2,n+2} = 0
\]

and, eliminating \( p \) and \( q \) between them, obtain the desired quadratic equation

\[
(40) \quad \begin{vmatrix}
(h/e)^2 & h/e & 1 \\
H_{n-1,n-1} & H_{n,n} & H_{n+1,n+1} \\
H_{n,n} & H_{n+1,n+1} & H_{n+2,n+2}
\end{vmatrix} = 0
\]

or

\[
(41) \quad \begin{vmatrix}
(z-a)^2/(u-a)^2 & (z-a)/(u-a) & f^2 \\
Q_{n-1,n-1} & Q_{n,n} & Q_{n+1,n+1} \\
Q_{n,n} & Q_{n+1,n+1} & Q_{n+2,n+2}
\end{vmatrix} = 0.
\]

Another pair of simple quadratic equations will be obtained by starting with

\[
(42) \quad \begin{align*}
m_1e_1/h_1^{n-1} + m_2e_2/h_2^{n-1} &= H_{n-1,k}, \\
m_1e_1/h_1^n + m_2e_2/h_2^n &= H_{n,k}, \\
m_1e_1/h_1^{n+1} + m_2e_2/h_2^{n+1} &= H_{n+1,k},
\end{align*}
\]

and

\[
(43) \quad \begin{align*}
m_1e_1/h_1^{k-1} + m_2e_2/h_2^{k-1} &= H_{n,k-1}, \\
m_1e_1/h_1^k + m_2e_2/h_2^k &= H_{n,k}, \\
m_1e_1/h_1^{k+1} + m_2e_2/h_2^{k+1} &= H_{n,k+1},
\end{align*}
\]

namely,

\[
(44) \quad \begin{vmatrix}
(z-a)^2 & (z-a)f & f^2 \\
Q_{n-1,k} & Q_{n,k} & Q_{n+1,k} \\
Q_{n,k} & Q_{n+1,k} & Q_{n+2,k}
\end{vmatrix} = 0
\]

and
The function \( Q_{n,k} = Q_{n,k}(z, u) \) may be evaluated by (3) or (4). However, it is more practical to use its expression in terms of \( D_j \) (see [3]),

\[
Q_{n,k} = \sum_{j=0}^{n-1} \binom{k}{j} (u - z)^{k-j} P_{n-1-j},
\]

which can be derived from (3) or (4) or in some other way.

If \( g(z, u) = (u - z)^k \), then

\[
Q_{n,k} = \sum_{j=0}^{n-1} \binom{k}{j} (u - z)^{k-j} P_{n-1-j}
\]

and the equations similar to (40), (43) and (44) can be derived in the same way.

The selection of the parameter \( u \) is not very difficult in many practical cases; however, it is a complicated problem in the general case.

For an illustration we again take the cubic trinomial \( z^3 + pz + q \). If its two zeros are imaginary, they are larger in absolute value than the real one when \( p > 0 \). Therefore, they cannot be calculated by (30), as in the previous illustration, but they can be calculated by (40), (43), or (44) with \( z = 0 \), as before, and \( u = -q/p \). It is a simple matter to show that the inequalities (36) are satisfied and the coefficients of the corresponding quadratic equations are expressible in terms of \( p \) and \( q \).

**References**