Theorem 1. A linear operator $T$ is compact if and only if it is a cluster point for the topology $\alpha$ of a sequence $\{F_n\}$ of continuous linear operators with finite dimensional range.

Remarks. To place the theorem in context consider all operators as mapping the Banach space $X$ into the Banach space $Y$ and consider the topology $\alpha$ to be that of almost uniform convergence on the unit ball of $X$ utilizing the norm topology on $Y$. The second theorem gives a similar result for weakly compact operators. All definitions and background information can be found in the two references.

Proof of Theorem 1. Assuming the sequence $\{F_n\}$ to have $T$ as a cluster point, there is a subnet $\{F_{\gamma}, \gamma \in G\}$ converging to $T$ for the topology $\alpha$. The second adjoint operators $\{F_{\gamma}^{**}, \gamma \in G\}$ form a Cauchy net for the topology of almost uniform convergence on the unit ball $S^{**}$ of $X^{**}$ utilizing the norm topology on $Y^{**}$ [1, Theorem 4.1]. Define a linear operator $P_0$ which agrees with $T^{**}$ on the image of $A$, while for all other $x^{**}$ in $X^{**}$, $P_0(x^{**}) = \lim_{\gamma} F_{\gamma}^{**}(x^{**})$.

Consider an arbitrary positive number $\varepsilon$ and a net $\{x^{**}_0, \gamma \in D\}$ in the image of $S$ in $X^{**}$ converging to a point $x^{**}_0$ in $X^{**}$ for the $X^*$ topology. There exist $\delta_0$ in $D$ and $\gamma_1, \gamma_2, \ldots, \gamma_k$ in $G$ such that

$$\min_{i=1, 2, \ldots, k} \|F_0(x^{**}_0) - F_{\gamma_i}^{**}(x^{**}_0)\| < \frac{\varepsilon}{3}$$

for all $\delta > \delta_0$, and

$$\|F_{\gamma_i}^{**}(x^{**}_0) - F_{\gamma_i}^{**}(x^{**})\| < \frac{\varepsilon}{3}$$

for $i=1, 2, \ldots, k$ and all $\delta > \delta_0$, and $\|F_{\gamma_i}^{**}(x^{**}_0) - F_0(x^{**}_0)\| < \varepsilon/3$ for $i=1, 2, \ldots, k$. Thus $\|F_0(x^{**}_0) - F_0(x^{**}_0)\| < \varepsilon$ for all $\delta > \delta_0$ and $F_0$ is continuous for the $X^*$ topology on $S^{**}$ and the norm topology on $Y^{**}$. Therefore $F_0$ is $T^{**}$ and $T$ is compact.

For the converse let $\mathcal{P}$ be the directed set composed of all continuous projections with finite dimensional range in $Y$, the order being determined by the inclusion ordering on their ranges. The net $\{P^{**}T^{**}, P \in \mathcal{P}\}$ converges pointwise to $T^{**}$ on $S^{**}$ and a known theorem for continuous functions says that due to the metric topology on $Y^{**}$ and the compactness of $S^{**}$ the net can be replaced by a

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sequence \( \{P_nT^*\} \) having \( T^* \) as a cluster point.\(^2\) \( T^* \) must also be a cluster point of the sequence for the topology of almost uniform convergence on \( S^* \) with the norm topology on \( Y^* \) [1, Theorem 4.2]. Therefore it is concluded that \( T \) is a cluster point of the sequence \( \{P_nT\} \) for the topology \( \alpha \).

By omitting the sequence and resorting to the topology \( \beta \), almost uniform convergence on the unit ball of \( X \) with the weak topology on \( Y \), the same line of reasoning gives Theorem 2.

**Theorem 2.** A linear operator \( T \) is weakly compact if and only if it is the limit point for the topology \( \beta \) of a net \( \{F_\gamma, \gamma \in G\} \) of continuous linear operators with finite dimensional range.

**References**


**University of Maryland**

\(^2\) The author is unable to locate a reference. A version of the theorem appeared at one time in a manuscript copy of J. L. Kelley's book on linear topological spaces. The statement is as follows. "Let \( F \) be a subset of the space of continuous functions with compact domain \( S \) and range in a metric space. If \( f \) is in the closure of \( F \) for the topology of pointwise convergence, then there is a sequence in \( F \) having \( f \) as a cluster point."