THE CIRCUMFERENCE OF A CONVEX POLYGON

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In this note we combine a convexity theorem due to Cauchy with a combinatorial identity discovered by M. Kac.

Cauchy's theorem [1] concerns the length $L$ of the circumference of a compact convex set $A$ in the plane. Let $D(\theta)$ denote the projection of $A$ on a line with direction $\theta$, $0 \leq \theta < \pi$, or, if $z = x + iy$

\begin{equation}
D(\theta) = \max_{x \in A} (x \cos \theta + y \sin \theta) - \min_{x \in A} (x \cos \theta + y \sin \theta).
\end{equation}

Then

\begin{equation}
L = \int_0^\pi D(\theta) d\theta.
\end{equation}

M. Kac, in [2], considered a vector $x = (x_1, x_2, \ldots, x_n)$ with real components. For each permutation

\[ \sigma = \{1, 2, \ldots, n\} \]

\[ \sigma_1 \sigma_2 \cdots \sigma_n \]

he defined the vectors

\[ x(\sigma) = (x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}), \]

and their partial sums

\[ s_0(\sigma) = 0, \quad s_k(\sigma) = x_{\sigma_1} + x_{\sigma_2} + \cdots + x_{\sigma_k}, \quad k = 1, 2, \ldots, n. \]

His result may be stated in the form

\begin{equation}
\sum_\sigma \left[ \max_{0 \leq k \leq n} s_k(\sigma) - \min_{0 \leq k \leq n} s_k(\sigma) \right] = \sum_\sigma \sum_{k=1}^n \frac{1}{k} \mid s_k(\sigma) \mid
\end{equation}

where the $\sigma$-summation extends over the group of all permutations of $n$-objects.

We shall consider a vector $z = (z_1, z_2, \ldots, z_n)$ with complex components. As above we let

\[ z(\sigma) = (z_{\sigma_1}, z_{\sigma_2}, \ldots, z_{\sigma_n}), \]

\[ s_0(\sigma) = 0, \quad s_k(\sigma) = z_{\sigma_1} + z_{\sigma_2} + \cdots + z_{\sigma_k}, \quad k = 1, 2, \ldots, n. \]

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We define the set \( A(\sigma) \) as the smallest convex set (polygon) containing all the points \( s_0(\sigma), s_1(\sigma), \cdots, s_n(\sigma) \), and \( L(\sigma) \) as the length of the circumference of \( A(\sigma) \). We shall obtain the following generalization of equation (3).

**Theorem 1.**

\[
(4) \quad \sum_\sigma L(\sigma) = 2 \sum_\sigma \sum_{k=1}^n \frac{1}{k} |s_k(\sigma)|.
\]

Here the summation again extends over all permutations, and (4) is equivalent to (3) when all \( z_k \) are real. When the \( z_k \) are not real we write

\[
z_k = x_k + iy_k, \quad z_{\sigma k} = x_{\sigma k} + iy_{\sigma k},
\]

\[
\ell_k(\theta) = x_k \cos \theta + y_k \sin \theta, \quad 0 \leq \theta < \pi,
\]

\[
u_0(\theta, \sigma) = 0, \quad \nu_k(\theta, \sigma) = \ell_{s_1}(\theta) + \cdots + \ell_{s_k}(\theta), \quad k = 1, 2, \cdots, n.
\]

Let \( D(\theta, \sigma) \) be the projection of \( A(\sigma) \) on a line with direction \( \theta \). Since \( A(\sigma) \) is the convex hull of its extreme points we have from (1)

\[
D(\theta, \sigma) = \max_{0 \leq k \leq n} \nu_k(\theta, \sigma) - \min_{0 \leq k \leq n} \nu_k(\theta, \sigma).
\]

By equation (2)

\[
L(\sigma) = \int_0^\pi \left[ \max_{0 \leq k \leq n} \nu_k(\theta, \sigma) - \min_{0 \leq k \leq n} \nu_k(\theta, \sigma) \right] d\theta.
\]

By equation (3)

\[
\sum_\sigma L(\sigma) = \sum_\sigma \sum_{k=1}^n \frac{1}{k} \int_0^\pi |\nu_k(\theta, \sigma)| d\theta.
\]

But

\[
\int_0^\pi |\nu_k(\theta, \sigma)| d\theta = 2 \left[ \left\{ \sum_{i=1}^k x_{s_i} \right\}^2 + \left\{ \sum_{i=1}^k y_{s_i} \right\}^2 \right]^{1/2} = |s_k(\sigma)|.
\]

Hence (4) is proved.

As an application we derive a result of probabilistic interest. Let \( Z_1, Z_2, \cdots \) denote a sequence of identically distributed independent complex valued random variables. Thus the distribution of each \( Z_k \) is the same planar Lebesgue-Stieltjes measure and their joint distributions are given by the obvious product measure. We define their partial sums as the random variables
Finally, let \( L_n \) be defined as the length of the circumference of the smallest convex set containing \( S_0, S_1, \cdots, S_n \). If \( "E" \) denotes expectation with respect to the product measure, we have

**Theorem 2.**

\[
E(L_n) = 2 \sum_{k=1}^{n} \frac{1}{k} E|S_k|.
\]

The proof is based on two observations. First, \( L_n \) is a continuous function of \( S_1, S_2, \cdots, S_n \), so that it is a random variable. Secondly, if we define the random variables \( Z_n, S_k(\sigma), L_n(\sigma) \) as was done in the deterministic case, it follows from the invariance of the product measure under permutations \( \sigma \) that the expectations \( E|S_n(\sigma)| \) and \( E(L_n(\sigma)) \) are independent of \( \sigma \). This proves the theorem and also shows that either both sides in (5) are finite or neither. Of course they are finite if and only if \( E|Z_i| < \infty \).

Finally, we consider two situations where the asymptotic behavior of \( E(L_n) \) is of some interest.

(a) Let \( Z_1, Z_2, \cdots \) be identically distributed and independent with

\[
Z_k = X_k + iY_k, \quad E(X_k) = E(Y_k) = 0, \quad E(X_k^2) = a^2 < \infty, \quad E(Y_k^2) = b^2 < \infty, \quad E(X_k Y_k) = \rho a b.
\]

Then \( n^{-1/2}(Z_1 + \cdots + Z_n) \) has a bivariate normal limiting distribution and its absolute value may be shown to be uniformly integrable in \( n \), so that

\[
\lim_{n \to \infty} n^{-1/2} E|Z_1 + \cdots + Z_n| = (2\pi)^{-1/2} \int_{0}^{\pi} [a^2 \sin^2 \theta + b^2 \cos^2 \theta + 2ab\rho \sin \theta \cos \theta]^{1/2} d\theta = c.
\]

It follows from Theorem 2 that

\[
\lim_{n \to \infty} n^{-1/2} E(L_n) = 4c.
\]

(b) Here we let \( X_1, X_2, \cdots \) be a sequence of identically distributed independent random variables with

\[
E(X_k) = \mu, \quad E[(X_k - \mu)^2] = \sigma^2 < \infty.
\]

We define the complex valued random variables
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\[ Z_k = X_k + i \]

and their partial sums

\[ S_0 = 0, \quad S_n = X_1 + \cdots + X_n + ni, \quad n = 1, 2, \ldots. \]

The law of large numbers asserts that \( n^{-1}S_n \to \mu + i \) with probability one. Geometrically this means that the polygonal path consisting of the points \( S_0, S_1, \ldots \) does not deviate too far from the straight line through 0 and \( \mu + i \). We shall denote by \( L_n \) the circumference of the smallest convex set containing the points \( S_0, S_1, \ldots, S_n \), and quite naturally, study the excess of \( L_n \) over its smallest possible value, which is \( 2n(1 + \mu^2)^{1/2} \).

Theorem 2 yields

\[ \frac{1}{2} E(L_n) - n(1 + \mu^2)^{1/2} = \sum_{k=1}^{n} \left[ \left\{ \frac{(X_1 + X_2 + \cdots + X_k)^2}{k} + 1 \right\}^{1/2} - (\mu^2 + 1)^{1/2} \right]. \]

Using the second order Taylor expansion of \((t^2+1)^{1/2}\) about \( t = \mu \), it is quite simple to show that, as \( k \to \infty \),

\[ E \left[ \left\{ \frac{(X_1 + \cdots + X_k)^2}{k} + 1 \right\}^{1/2} - (\mu^2 + 1)^{1/2} \right] \sim \frac{1}{2} (1 + \mu^2)^{-3/2} E \left[ \left( \frac{X_1 + \cdots + X_k - \mu}{k} \right)^2 \right] = \frac{1}{2} (1 + \mu^2)^{-3/2} \sigma^2. \]

It follows that

\[ \lim_{n \to \infty} (\log n)^{-1}[E(L_n) - 2n(1 + \mu^2)^{1/2}] = \sigma^2(1 + \mu^2)^{-3/2}. \]

REFERENCES


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