SEQUENCES OF HOMEOMORPHISMS ON THE $n$-SPHERE

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1. Introduction. In his recent book of mathematical problems, S. Ulam (see [1, p. 46]) states the following question as one which he and Borsuk have considered:

Given an arbitrary closed subset $C$ of an $n$-sphere $S$, $n>0$, does there exist a sequence $H_1, H_2, H_3, \cdots$ of homeomorphisms of $S$ onto itself such that for every $p$ of $S$, $\lim_{k \to \infty} H_k(p) \exists$ and is in $C$, and every point of $C$ is such a limit?

This problem also occurs in the original Scottish Book, along with the remark that Borsuk has solved the problem for the case in which $S$ is two-dimensional.

In this note an affirmative answer is obtained for the above question in the general case. The proof leans heavily upon a result which the author obtained in [2].

2. Admissible polyhedra. Let $\Sigma$ be the set of all closed $n$-cubes which are contained in the euclidean $n$-space $R^n$ and whose edges are parallel to the coordinate axes.

If $J \subset \Sigma$, a subset $A$ of the boundary of $J$ is an $\alpha$-set of $J$ if $A$ is the union of a collection of $(n-1)$-dimensional faces of $J$ and for some such face $\sigma$, $\sigma$ is contained in $A$ while the $(n-1)$-dimensional face opposite $\sigma$ is not contained in $A$.

A polyhedron $P$ is admissible if there exists a sequence $P_1, \cdots, P_k$ of polyhedra such that: $P_1 \subset \Sigma$, $P_k = P$, and for each $i = 1, \cdots, k-1$, $P_{i+1} = P_i \cup J_i$ where $J_i \subset \Sigma$ and $P_i \cap J_i$ is an $\alpha$-set of $J_i$.

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Lemma 1. If \( P \) is an admissible polyhedron, \( J \subseteq \Sigma \), \( P \cap J \) is an \( \alpha \)-set of \( J \), and \( U \) and \( V \) are open sets which contain \( J \) and \( P \) respectively, then there exists a homeomorphism \( h \) of \( R_n \) onto itself such that \( h[P \cup J] \subseteq V \) and \( h \) is the identity on \( R^n - U \).

Proof. There is an \((n-1)\)-dimensional face \( \sigma \) of \( J \) such that \( \sigma \subseteq P \cap J \) and \( \sigma' \subseteq P \cap J \), where \( \sigma' \) is the face of \( J \) which is opposite \( \sigma \). We may assume without loss of generality that \( R^n = R \times R^{n-1} \) and that \( J \) is situated so that \( \sigma \subseteq \{0\} \times R^{n-1} \) and \( \sigma' \subseteq \{1\} \times R^{n-1} \). There exists \( \epsilon > 0 \), a closed set \( M \) which is contained in the projection of \( \sigma \) onto \( R^{n-1} \), and an open set \( W \supseteq M \) in \( R^{n-1} \) such that \( [\epsilon, 1] \times M \subset J - V \), \((0, 1+\epsilon) \times W \) is disjoint from \( P \), and \((0, 1+\epsilon) \times W \subset U \).

By Urysohn’s lemma, there exists a continuous function \( \phi \) on \( R^{n-1} \) into \([\epsilon, 1]\) such that \( \phi(x) = \epsilon \) for \( x \in M \) and \( \phi(x) = 1 \) for \( x \in R^{n-1} - W \).

We now define \( h \) as follows: \( h \) is the identity on \( R^n - (0, 1+\epsilon) \times W \), and if \( x \in W \) then \( h \) maps the segment from \((0, x)\) to \((1, x)\) linearly onto the segment from \((0, x)\) to \((\phi(x), x)\) and \( h \) maps the segment from \((1, x)\) to \((1+\epsilon, x)\) linearly onto the segment from \((\phi(x), x)\) to \((1+\epsilon, x)\). It is easy to see that \( h \) has the desired properties.

Lemma 2. If \( G \) is a connected, open subset of \( R^n \), \( P \subseteq G \) is an admissible polyhedron, and \( Q \subseteq G \) is a nonempty open set, then there exists a homeomorphism \( H \) of \( R^n \) onto \( R^n \) such that \( H[P] \subseteq Q \) and \( H \) is the identity on \( R^n - G \).

Proof. There exists a sequence \( P_1, \ldots, P_k \) of admissible polyhedra such that \( P_i \subseteq \Sigma \), \( P_k = P \), and for each \( i = 1, \ldots, k-1 \), we have \( P_{i+1} = P_i \cup J_i \) where \( J_i \subseteq \Sigma \) and \( P_i \cap J_i \) is an \( \alpha \)-set of \( J_i \). It is obvious that there is a homeomorphism \( h_1 \) of \( R^n \) onto \( R^n \) such that \( h_1 \) is the identity outside \( G \) and \( h_1[P_i] \subseteq Q \). Because of continuity of \( h_1 \), there is an open set \( V_1 \supseteq P_1 \) such that \( h_1[V_1] \subseteq Q \). By Lemma 1, there is a homeomorphism \( h_2 \) of \( R^n \) onto \( R^n \) such that \( h_2 \) is the identity outside \( G \) and \( h_2[P_2] \subseteq V_1 \), and by continuity of \( h_2 \) there is an open set \( V_2 \supseteq P_2 \) such that \( h_2[V_2] \subseteq V_1 \). Continuing in this manner, we obtain open sets \( V_i \supseteq P_i \) for each \( i \), and homeomorphisms \( h_i \) of \( R^n \) onto \( R^n \) which are the identity outside \( G \) and which satisfy \( h_{i+1}[V_{i+1}] \subseteq V_i \). The composite homeomorphism \( H = h_1 h_2 \cdots h_k \) has the desired properties.

3. The main result.

Theorem. Let \( S \) be an \( n \)-sphere, \( n > 0 \), and let \( C \) be a nonempty closed subset of \( S \). Then there exists a sequence \( H_1, H_2, H_3, \ldots \) of homeomor-
homeomorphisms of $S$ onto itself such that $H_k(p) = p$ for each $p \in C$ and each positive integer $k$, and $\lim_{k \to \infty} H_k(p)$ exists and is in $C$ for each $p \in S - C$.

Proof. Choose a point $q \in C$ and let $\Phi$ be a homeomorphism of $S - \{q\}$ onto euclidean $n$-space $\mathbb{R}^n$. We define $C^* = \Phi[C - \{q\}]$. We are going to define a sequence $h_1, h_2, h_3, \cdots$ of homeomorphisms of $\mathbb{R}^n$ onto itself.

We enumerate the components of $\mathbb{R}^n - C^*$ in a sequence $G_1, G_2, G_3, \cdots$ and choose a point $p_k$ in the boundary of $G_k$ for each positive integer $k$. (We assume that $\mathbb{R}^n - C^*$ has an infinite number of components, since the case in which $\mathbb{R}^n - C^*$ has a finite number of components can be handled in a similar manner.)

It follows from the theorem proved in [2] that there exist sets $P(k, j)$ for $k$ and $j$ positive integers such that: each set $P(k, j)$ is a closed topological $n$-cell, $P(k, 1) \subset P(k, 2) \subset P(k, 3) \subset \cdots$ for each $k$, and $G_k = \bigcup_{j=1}^{\infty} P(k, j)$ for each $k$. It follows from the construction used in [2] that the sets $P(k, j)$ may be chosen so as to be admissible polyhedra, and we assume that this has been done. We choose non-empty open sets $Q(k, j)$ for all positive integers $k$ and $j$ so that $Q(k, j) \subset G_k$ and $\lim_{j \to \infty} Q(k, j) = \{p_k\}$. It is now easy to use Lemma 2 to construct for each positive integer $k$ a homeomorphism $h_k$ of $\mathbb{R}^n$ onto itself such that: $h_k[P(i, k)] \subset Q(i, k)$ for $1 \leq i \leq k$, and $h_k$ is the identity on $\mathbb{R}^n - \bigcup_{i=1}^{k} G_i$. It is easy to see that $h_j(x) = x$ for all $x \in C^*$ and all $j$, and that $\lim_{j \to \infty} h_j(x) = p_k \in C^*$ for $x \in G_k$.

The desired homeomorphisms of $S$ onto $S$ are now obtained by defining

$$H_k(p) = \begin{cases} q & \text{for } p = q, \\ \Phi^{-1} h_k \Phi(p) & \text{for } p \in S - \{q\}. \end{cases}$$

Bibliography


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