1. Introduction. In his recent book of mathematical problems, S. Ulam (see [1, p. 46]) states the following question as one which he and Borsuk have considered:

**Given an arbitrary closed subset C of an n-sphere S, n > 0, does there exist a sequence H₁, H₂, H₃, · · · of homeomorphisms of S onto itself such that for every p of S, \( \lim_{k \to \infty} H_k(p) \) exists and is in C, and every point of C is such a limit?**

This problem also occurs in the original Scottish Book, along with the remark that Borsuk has solved the problem for the case in which S is two-dimensional.

In this note an affirmative answer is obtained for the above question in the general case. The proof leans heavily upon a result which the author obtained in [2].

2. Admissible polyhedra. Let \( \Sigma \) be the set of all closed n-cubes which are contained in the euclidean n-space \( R^n \) and whose edges are parallel to the coordinate axes.

If \( J \subseteq \Sigma \), a subset \( A \) of the boundary of \( J \) is an \( \alpha \)-set of \( J \) if \( A \) is the union of a collection of \( (n-1) \)-dimensional faces of \( J \) and for some such face \( \sigma \), \( \sigma \) is contained in \( A \) while the \( (n-1) \)-dimensional face opposite \( \sigma \) is not contained in \( A \).

A polyhedron \( P \) is **admissible** if there exists a sequence \( P_1, \ldots, P_k \) of polyhedra such that: \( P_1 \subseteq \Sigma, P_k = P \), and for each \( i = 1, \ldots, k-1 \), \( P_{i+1} = P_i \cup J_i \) where \( J_i \subseteq \Sigma \) and \( P_i \cap J_i \) is an \( \alpha \)-set of \( J_i \).

---

1 The author is an Alfred P. Sloan Research Fellow.
Lemma 1. If $P$ is an admissible polyhedron, $J \subseteq \Sigma$, $P \cap J$ is an $\alpha$-set of $J$, and $U$ and $V$ are open sets which contain $J$ and $P$ respectively, then there exists a homeomorphism $h$ of $\mathbb{R}^n$ onto itself such that $h[P \cup J] \subseteq V$ and $h$ is the identity on $\mathbb{R}^n - U$.

Proof. There is an $(n-1)$-dimensional face $\sigma$ of $J$ such that $\sigma \subseteq P \cap J$ and $\sigma' \subseteq P \cap J$, where $\sigma'$ is the face of $J$ which is opposite $\sigma$. We may assume without loss of generality that $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ and that $J$ is situated so that $\sigma \subseteq \{0\} \times \mathbb{R}^{n-1}$ and $\sigma' \subseteq \{1\} \times \mathbb{R}^{n-1}$. There exists $\varepsilon > 0$, a closed set $M$ which is contained in the projection of $\sigma$ onto $\mathbb{R}^{n-1}$, and an open set $W \supseteq M$ in $\mathbb{R}^{n-1}$ such that $[\varepsilon, 1] \times M \cap J - V$, $(0, 1+\varepsilon) \times W$ is disjoint from $P$, and $(0, 1+\varepsilon) \times W \subseteq U$.

By Urysohn's lemma, there exists a continuous function $\phi$ on $\mathbb{R}^{n-1}$ into $[\varepsilon, 1]$ such that $\phi(x) = \varepsilon$ for $x \in M$ and $\phi(x) = 1$ for $x \in \mathbb{R}^{n-1} - W$.

We now define $h$ as follows: $h$ is the identity on $\mathbb{R}^n - (0, 1+\varepsilon) \times W$, and if $x \in W$ then $h$ maps the segment from $(0, x)$ to $(1, x)$ linearly onto the segment from $(0, x)$ to $(\phi(x), x)$ and $h$ maps the segment from $(1, x)$ to $(1+\varepsilon, x)$ linearly onto the segment from $(\phi(x), x)$ to $(1+\varepsilon, x)$. It is easy to see that $h$ has the desired properties.

Lemma 2. If $G$ is a connected, open subset of $\mathbb{R}^n$, $P \subseteq G$ is an admissible polyhedron, and $Q \subseteq G$ is a nonempty open set, then there exists a homeomorphism $H$ of $\mathbb{R}^n$ onto $\mathbb{R}^n$ such that $H[P] \subseteq Q$ and $H$ is the identity on $\mathbb{R}^n - G$.

Proof. There exists a sequence $P_1, \ldots, P_k$ of admissible polyhedra such that $P_1 \subseteq \Sigma$, $P_k = P$, and for each $i = 1, \ldots, k-1$, we have $P_{i+1} = P_i \cup J_i$ where $J_i \subseteq \Sigma$ and $P_i \cap J_i$ is an $\alpha$-set of $J_i$. It is obvious that there is a homeomorphism $h_1$ of $\mathbb{R}^n$ onto $\mathbb{R}^n$ such that $h_1$ is the identity outside $G$ and $h_1[P_1] \subseteq Q$. Because of continuity of $h_1$, there is an open set $V_1 \supseteq P_1$ such that $h_1[V_1] \subseteq Q$. By Lemma 1, there is a homeomorphism $h_2$ of $\mathbb{R}^n$ onto $\mathbb{R}^n$ such that $h_2$ is the identity outside $G$ and $h_2[P_2] \subseteq V_1$, and by continuity of $h_2$ there is an open set $V_2 \supseteq P_2$ such that $h_2[V_2] \subseteq V_1$. Continuing in this manner, we obtain open sets $V_i \supseteq P_i$ for each $i$, and homeomorphisms $h_i$ of $\mathbb{R}^n$ onto $\mathbb{R}^n$ which are the identity outside $G$ and which satisfy $h_{i+1}[V_{i+1}] \subseteq V_i$. The composite homeomorphism $H = h_1 h_2 \cdots h_k$ has the desired properties.

3. The main result.

Theorem. Let $S$ be an $n$-sphere, $n > 0$, and let $C$ be a nonempty closed subset of $S$. Then there exists a sequence $H_1, H_2, H_3, \ldots$ of homeomor-
 Sequences of Homeomorphisms on the n-Sphere

Sequences of homeomorphisms on $S$ onto itself such that $H_k(p) = p$ for each $p \in C$ and each positive integer $k$, and $\lim_{k \to \infty} H_k(p)$ exists and is in $C$ for each $p \in S - C$.

Proof. Choose a point $q \in C$ and let $\Phi$ be a homeomorphism of $S - \{q\}$ onto euclidean $n$-space $R^n$. We define $C^* = \Phi[C - \{q\}]$. We are going to define a sequence $h_1, h_2, h_3, \ldots$ of homeomorphisms of $R^n$ onto itself.

We enumerate the components of $R^n - C^*$ in a sequence $G_1, G_2, G_3, \ldots$ and choose a point $p_k$ in the boundary of $G_k$ for each positive integer $k$. (We assume that $R^n - C^*$ has an infinite number of components, since the case in which $R^n - C^*$ has a finite number of components can be handled in a similar manner.)

It follows from the theorem proved in [2] that there exist sets $P(k, j)$ for $k$ and $j$ positive integers such that: each set $P(k, j)$ is a closed topological $n$-cell, $P(k, 1) \subset P(k, 2) \subset P(k, 3) \subset \cdots$ for each $k$, and $G_k = \bigcup_{j=1}^{\infty} P(k, j)$ for each $k$. It follows from the construction used in [2] that the sets $P(k, j)$ may be chosen so as to be admissible polyhedra, and we assume that this has been done. We choose non-empty open sets $Q(k, j)$ for all positive integers $k$ and $j$ so that $Q(k, j) \subset G_k$ and $\lim_{j \to \infty} Q(k, j) = \{p_k\}$. It is now easy to use Lemma 2 to construct for each positive integer $k$ a homeomorphism $h_k$ of $R^n$ onto itself such that: $h_k[P(i, k)] \subset Q(i, k)$ for $1 \leq i \leq k$, and $h_k$ is the identity on $R^n - \bigcup_{i=1}^{k} G_i$. It is easy to see that $h_j(x) = x$ for all $x \in C^*$ and all $j$, and that $\lim_{j \to \infty} h_j(x) = p_k \in C^*$ for $x \in G_k$.

The desired homeomorphisms of $S$ onto $S$ are now obtained by defining

$$H_k(p) = \begin{cases} q & \text{for } p = q, \\ \Phi^{-1} h_k \Phi(p) & \text{for } p \in S - \{q\}. \end{cases}$$

Bibliography


The University of Georgia