A BANACH SPACE CHARACTERIZATION OF
PURELY ATOMIC MEASURE SPACES

R. R. PHELPS

It is well known [4, p. 265; 3] that the space $L_1[0, 1]$ is not isomorphic with a conjugate space. At the other extreme, it is also well known that $l_1$ is isometric with the conjugate space of $c_0$. Each of these is an example of a space of all real-valued integrable functions over a measure space $(T, \mu)$, a major difference between them being that the measure space underlying $L_1[0, 1]$ has no atoms, while that underlying $l_1$ is purely atomic. It is natural to conjecture that a space $L_1(T, \mu)$ is isomorphic with a conjugate space if and only if $(T, \mu)$ is purely atomic; we will show that this conjecture is false, although it is true for separable $L_1$ spaces. We prove this result, together with one of our characterizations of purely atomic $(T, \mu)$, by using the notion of differentiability of vector-valued functions of bounded variation on $[0, 1]$. (This was the method employed by Gelfand [4] in proving the result cited above.) A related result is given in terms of locally uniformly convex spaces [8].

Let $(T, \mu)$ be a measure space. (We do not assume that $T$ is measurable.) An atom $A \subset T$ is a measurable set such that $0 < \mu(A) < \infty$, and for each measurable set $B \subset A$, either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. We will consider two atoms to be the "same" if they differ by a set of measure zero. A set $S$ of positive finite measure is purely atomic if the set $S = \bigcup \{A \subset S: A$ is an atom$\}$ has measure zero. (Since atoms are essentially disjoint, $\mu$ is countably additive, and $\mu(S) < \infty$, $S$ can contain at most countably many atoms, and hence the above set is measurable.) We say that the measure space $(T, \mu)$ is purely atomic if every subset $S \subset T$ of positive finite measure is purely atomic. We denote by $\mathcal{A}$ the collection of all atoms $A \subset T$. There are doubtless other possible definitions of "purely atomic"; that the one given here is reasonable is shown by the following lemma.

**Lemma.** If $(T, \mu)$ is purely atomic, then $L_1(T, \mu)$ is isometric with $l(\mathcal{A})$ (and hence is a conjugate space).

**Proof.** The space $l(\mathcal{A})$ is the set of all real functions $y$ on $\mathcal{A}$ such that $\|y\| = \sum_{A \in \mathcal{A}} |y(A)|$ is finite, the summation being taken over the directed system of all finite subsets of $\mathcal{A}$. (See [2] for a proof that $l_1(\mathcal{A})$ is isometric with the conjugate space of $c_0(\mathcal{A})$.) We first show

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that any $\sigma$-finite subset $S$ of $\mathbb{R}$ is purely atomic, i.e., if $S = \bigcup_{i=1}^{\infty} S_i$, where the $S_i$ are pairwise disjoint sets of positive finite measure, then (letting $U$ be the union of all the atoms in $S$) we have $\mu(S \sim U) = 0$. Indeed, each atom of $S$ is contained in some $S_i$; since each $S_i$ contains at most countably many atoms, the same is true of $S$ and therefore $S \sim U$ is measurable. If $S \sim U$ were to have positive measure, the equality $S \sim U = \bigcup (S \sim U) \cap S_i$ would imply that at least one set $(S \sim U) \cap S_i$ would have positive measure, and would therefore contain an atom $A$. Since $A$ would also be an atom in $S$, but not in $U$, this would be a contradiction.

Now, if $x \in L_1(T, \mu)$ and $A$ is an atom, then $x$ is constant a.e. on $A$. Let $(\phi x)(A) = x(A)\mu(A)$; then $\sum_A |(\phi x)(A)| = \sum_A |x(A)| \mu(A) \leq \int_T |x| \, d\mu < \infty$, so $\phi x \in L_1(\mathcal{A})$. If $y \in L_1(\mathcal{A})$, then the element $x$ which equals $y(A)\mu(A)^{-1}$ on each atom and is zero elsewhere is in $L_1(T, \mu)$ and hence $\phi$ is onto. Since $\phi$ is clearly linear, we need only show that it is an isometry, i.e. that for each $x \in L_1(T, \mu)$, $\sum_A |x(A)| \mu(A) = \int_T |x| \, d\mu$.

Let $S(x) = \{ t \in T : x(t) \neq 0 \}$; this set is easily seen to be $\sigma$-finite and therefore purely atomic. Now (letting $U(x)$ be the union of all the atoms in $S(x)$), $\int_T |x| \, d\mu = \sum_A |x(A)| \mu(A) + \int_{S(x) \sim U(x)} |x| \, d\mu$; since $S(x)$ is purely atomic, the second term is zero and $\phi$ is an isometry.

A Banach space $E$ is isomorphic with a Banach space $F$ if there exists a continuous, linear one-to-one map of $E$ onto $F$ which has a continuous inverse. The existence of a Banach space isomorphic to $E$ is equivalent to the existence of positive constants $k$ and $K$ and norms $\| \cdot \|$ and $\| | \cdot | |$ on $E$ such that $k\|x\| \leq \|x\| \leq K\|x\|$ for all $x \in E$.

A normed space $E$ is locally uniformly convex if for each $x \in E$ such that $\|x\| = 1$, and for each $\varepsilon > 0$, there exists $\delta(x, \varepsilon) > 0$ such that $\|x + y\| \leq 2 - \delta$ whenever $\|x - y\| \geq \varepsilon$. It is easily seen that uniform convexity [2] implies local uniform convexity, and the latter implies strict convexity; Lovaglia [8] shows that neither of these implications may be reversed. An equivalent formulation in terms of sequences ($\|x\| = 1 = \|y_n\|$ and $\|x + y_n\| \rightarrow 2$ imply $\|x - y_n\| \rightarrow 0$) shows that $E$ is locally uniformly convex if and only if each separable subspace of $E$ is locally uniformly convex.

The following theorem has been proved by Lovaglia [8, Theorem 3.1] in a more general context, but in a slightly different way. The adaptation of his proof to this special case is shorter; more importantly, our method of renorming $l_1$ will enable us to apply the result to nonseparable $l_1$ spaces.
LOVAGLIA'S Theorem. The space $l_1$ is isomorphic with a locally uniformly convex space.

Proof. By $l_1$ we mean, of course, the space of all sequences $x$ such that $\|x\| = \sum |x_i| < \infty$. Define a new norm of $l_1$ by

$$\|x\|_1 = (\|x\|^2 + \sum x_i^2)^{1/2};$$

it is easily checked that $\|x\| \leq \|x\|_1 \leq (2)^{1/2}\|x\|$. To see that this norm makes $l_1$ locally uniformly convex, suppose that $\|x\|_1 = \|\gamma^n\|_1$ and $\|x + \gamma^n\|_1 \to 2$, but $\lim \|x - \gamma^n\|_1$ (and hence $\lim \|x - \gamma^n\|$) $\neq 0$. Then there exists a subsequence of the $\gamma$'s (say $\{\gamma^n\}$) and $t > 0$ such that $\|x - \gamma^n\| \geq t$. Since $\|\gamma^n\| \leq 1$ and $|\gamma^n_i| \leq 1$ for each $i$, we can use the diagonal process to obtain a subsequence such that $\|\gamma^n\| \to a$, say, while $\gamma^n_i \to a_i$ for each $i$. Thus,

$$\lim \sum_{k=1}^{\infty} (\gamma^n_i)^2 = \lim \left(1 - \sum_{i}^{k} (\gamma^n_i)^2 - \|\gamma^n\|^2\right) = 1 - \sum_{i}^{k} a_i^2 - a^2 = b_k^2,$$

say. Now, for each $k \geq 1$ we have

$$\|x + \gamma^n\|_1$$

$$\leq \left\{ \sum_{i}^{k} (x_i + \gamma^n_i)^2 + (\|x\| + \|\gamma^n\|)^2 + \left[ \sum_{k=1}^{\infty} (x_i + \gamma^n_i)^2 \right]^{(1/2)^2}\right\}^{1/2}$$

$$\leq \left\{ \sum_{i}^{k} (x_i + \gamma^n_i)^2 + (\|x\| + \|\gamma^n\|)^2$$

$$+ \left[ \left( \sum_{k=1}^{\infty} x_i^2 \right)^{1/2} + \left( \sum_{k=1}^{\infty} (\gamma^n_i)^2 \right)^{1/2}\right]^{2}\right\}^{1/2},$$

the latter being obtained by Minkowski's inequality. Taking limits as $n \to \infty$ and applying the Minkowski inequality once again yields

$$2 = \lim \|x + \gamma^n\|_1$$

$$\leq \left\{ \sum_{i}^{k} (x_i + a_i)^2 + (\|x\| + a)^2 + \left( \sum_{k=1}^{\infty} x_i^2 \right)^{1/2} + b_k \right\}^{1/2}$$

$$\leq \left[ \sum_{i}^{k} x_i^2 + \|x\|^2 + \sum_{k=1}^{\infty} x_i^2 \right]^{1/2} + \left[ \sum_{i}^{k} a_i^2 + a^2 + b_k^2 \right]^{1/2} = 2.$$

Since equality holds throughout (for all $k \geq 1$), it follows that $\|x\| = a$ and $x_i = a_i$ for all $i$. We see, then, that $t \leq \|x - \gamma^n\| \leq \sum_{i}^{k} |x_i - \gamma^n_i| + \sum_{k=1}^{\infty} |\gamma^n_i|$, so for $k \geq 1$,
If we choose \( k \geq k_0 \), say, the right side will be no less than \( u > 0 \); if \( k_0 \) is sufficiently large, we will have \( 0 < \sum_i^k \| x_i \| < 1 \) and hence, for \( k \geq k_0 \),

\[
\lim \inf \sum_{k+1}^{\infty} \| y^n_i \| \left( \sum_{1}^{k} \| x_i \| \right)^{-1} \geq u \left( \sum_{1}^{k} \| x_i \| \right)^{-1} > u > 0.
\]

Choosing \( K \geq k_0 \) such that \( (1 + u) \sum_{1}^{K} \| x_i \| > \| x \| \), we see that there exists a subsequence of the \( y \)'s such that

\[
\sum_{K+1}^{\infty} \| y^n_i \| \left( \sum_{1}^{K} \| x_i \| \right)^{-1} > u > 0.
\]

Finally, then, we have

\[
\sum_{1}^{K} \| y^n_i \| \left( \sum_{1}^{K} \| y^n_i \| \right)^{-1} + \sum_{K+1}^{\infty} \| y^n_i \| \left( \sum_{1}^{K} \| x_i \| \right)^{-1} > 1 + u
\]

so that \( \lim \inf \| y^n \| (\sum_{1}^{K} \| x_i \|)^{-1} \geq 1 + u \). Since \( \lim \inf \| y^n \| = \lim \| y^n \| = \| x \| \), this contradicts the inequality used in defining \( K \), and the proof is complete.

A function \( \phi \) defined on \([0, 1]\) whose range lies in a normed space \( E \) is of bounded variation if \( \sup \sum |\phi(r_{i+1}) - \phi(r_i)| < \infty \), where the supremum is taken over all partitions \( 0 = r_0 < r_1 < \cdots < r_n = 1 \) of \([0, 1]\). We say that \( \phi \) is differentiable a.e. if the limit

\[
\phi'(r) = \lim_{h \to 0} \left[ \phi(r + h) - \phi(r) \right] h^{-1}
\]

exists for all \( r \) in \([0, 1]\) outside a set of Lebesgue measure zero. The space \( E \) has property (D) if every \( \phi \) from \([0, 1]\) into \( E \) which is of bounded variation is differentiable a.e. Note that \( E \) has property (D) if and only if \( E \) has property (D) under an equivalent norm, i.e. property (D) is "preserved" under isomorphism.

We now state a theorem concerning property (D) which will be of use in what follows.

**Gelfand's Theorem.** If a separable Banach space \( E \) is isomorphic with a conjugate space, then \( E \) has property (D).

Gelfand proved this in [4, p. 264]; an interesting proof has also been given by Alaoglu in [1].
(i) \((T, \mu)\) is purely atomic.

(ii) \(L_1(T, \mu)\) is isomorphic with a locally uniformly convex Banach space.

(iii) \(L_1(T, \mu)\) has property (D).

Proof. (i) implies (ii). If \((T, \mu)\) is purely atomic, we may assume, by virtue of the above lemma, that \(L_1(T, \mu)\) is \(l_1(S)\) for some set \(S\). Define a new norm on \(l_1\) by \(\|x\|_2 = (\sum |x(s)|^2 + \sum x(s)^2)^{1/2}\). We need only show that, under this norm, every separable subspace (and hence \(l_1(S)\) itself) is locally uniformly convex. Let \(M\) be any separable subspace of \(l_1(S)\) and let \(\{x_n\}_1^\infty\) be a dense sequence in \(M\). As in the proof of the lemma, the support \(S(x_n)\) of each \(x_n\) is countable, so the set \(S_M = \bigcup_{n=1}^\infty S(x_n)\) is also countable. Since \(S(x) \subset S_M\) for each \(x \in M\), we see that \(M \subset l_1(S_M)\), the (separable) subspace of all elements in \(l_1(S)\) which vanish outside \(S_M\). But, by our proof of Lovaglia’s theorem, \(l_1(S_M)\) is locally uniformly convex under the norm induced by \(\| \cdot \|_1\).

(iii) implies (i). Suppose that \((T, \mu)\) is not purely atomic. Then \(T\) contains a subset \(S'\) of finite positive measure which is not the union of atoms; letting \(S = S' \cup \bigcup \{A : A\) is an atom, \(A \subset S'\}\), we see that \(0 < \mu(S) < \infty\) and \(S\) contains no atoms. In the terminology of [5], the restriction of \(\mu\) to the measurable subsets of \(S\) is a convex measure, i.e. there exists a measurable function \(f\) defined on \(S\) with range \([0, 1]\) such that \(\mu\{s \in S : f(s) < r\} = r\mu(S)\) for each \(0 \leq r \leq 1\). Define \(\phi\) on \([0, 1]\) by letting \(\phi(r)\) be the characteristic function of \(\{s \in S : f(s) < r\}\). Then \(\phi(r) \in L_1(T, \mu)\) and \(\|\phi(r) - \phi(r')\| = |r - r'|\mu(S)\), so \(\phi\) is of bounded variation. It is not differentiable at any point of \([0, 1]\), however, since if \(0 < r < 1\), let \(0 < h < \min(r, 1 - r)\) and verify that \(\|\phi(r + h) - \phi(r)\| = h^{-1} - \|\phi(r - h) - \phi(r)\| = 2\mu(S)\). Thus, \(L_1(T, \mu)\) does not have property (D), and the proof is complete.

(i) implies (ii). We simply observe that a function of bounded variation has at most countably many discontinuities, so that its range lies in a separable subspace \(M\) of \(l_1(S)\). Since \(M \subset l_1(S_M)\) and the latter is a separable conjugate space, Gelfand’s theorem applies.

Corollary. Suppose that \(L_1(T, \mu)\) is separable. Then \((T, \mu)\) is purely atomic if and only if \(L_1(T, \mu)\) is isomorphic with a conjugate space.

Proof. By the lemma, we see that if \((T, \mu)\) is purely atomic, then \(L_1(T, \mu)\) is isometric with a conjugate space. The Gelfand theorem and “(iii) implies (i)” of the above theorem prove the converse.

By a result of Kakutani [6], the second conjugate \(E\) of \(L_1[0, 1]\) is an abstract \((L)\)-space and hence [7] is of the form \(L_1(T, \mu)\) for some...
measure space \((T, \mu)\). Since Lebesgue measure on \([0, 1]\) is nonatomic, our theorem shows that \(L_1[0, 1]\) does not have property (D), and hence (using the natural embedding of a Banach space into its second conjugate) \(E\) does not have property (D). By the theorem again, \(E\) is not purely atomic, i.e. there exists a measure space \((T, \mu)\), which is not purely atomic, such that \(L_1(T, \mu)\) is a conjugate space.

The problem posed by Dieudonné in [3] remains open: Characterize those \((T, \mu)\) for which \(L_1(T, \mu)\) is isometric (or isomorphic) with a conjugate space.

*Added in proof.* M. I. Kadec [Izvestia Vyših Učebnych Zavedenii. Mat. vol. 6 (13) (1959) pp. 51–57] has proved the interesting fact that every separable Banach space is isomorphic with a locally uniformly convex space.

**Bibliography**


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