ERRATUM TO A NEW INEQUALITY FOR THE GREEN'S FUNCTION

WILLIAM C. FOX

The paper [1] gave a short elementary derivation of the best-possible value of Koebe's constant, a derivation using geometric methods free of the arithmetic reasoning characteristic of "hard analysis." Professor Albert Pfluger has informed me that the inequality in my Lemma 3.1 must be reversed. This note contains a revised version of that lemma and of the proof of Theorem 3.2, which proof depends upon that lemma.

Lemma (3.1)* (The preliminary inequality). The simple closed curve \( J_r \) in the upper half plane which is mapped by the function \( F_r(w) = w^2 + r \) onto the circle whose radius is \( r > 0 \) and whose center is the origin separates the sphere into two domains: \( E_r \), the exterior one, and \( I_r \), the interior one. If \( e \) is any point of \( E_r \), then \( E_r \) admits a Green's function \( U_r \) (with pole at \( e \)) which is continuous on \( (E_r \cup I_r) - \{ e \} \) and which vanishes on \( J_r \). Moreover, if \(-e\) is in \( I_r \) and if \( 0 < r < 1 \) then the inequality

\[
U_r(w) \leq \ln \left| \frac{w/(r)^{1/2} + e}{w - e} \right|
\]

holds for all \( w \) in \( E_r \).

In [1], Lemma 3.1 claimed that

\[
U_r(w) \leq \ln \left| \frac{w + e}{w - e} \right|.
\]

Since the right-hand member is the Green's function of that domain which contains \( e \) and is complementary to the line \( |w + e| = |w - e| \) and since that domain is contained within \( E_r \) when, e.g., \( e = -i \) and \( r = 1 \) (the case used in Theorem 3.2), the lemma's claim was false.

Proof of (3.1)*. The existence of \( U_r \) and its property of vanishing on \( J_r \) are proved in [1]. The rest of the proof of (3.1)* proceeds as follows. Let \( G_r \) be defined by the rule

\[
G_r(w) = \frac{w - e}{w/(r)^{1/2} + e}.
\]

Evidently \( G_r(e) = 0 \) and \( G_r(-er^{1/2}) = \infty \), so that \( \ln |G_r| \) is harmonic.

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on the sphere except at the points \( e \) and \(-er^{1/2}\). Therefore the function \( H_r = U_r + \ln |G_r| \) is harmonic on \( E_r \) except possibly at \( e \) and \(-er^{1/2}\). By definition of the Green's function \( H_r \) is harmonic at \( e \). Choosing \(-e\) in \( I_r \) (whence \( |F_r(e)| = |e^2 + 1| < 1 \)) and choosing \( 0 < r < 1 \), it follows that \( |F_r(-er^{1/2})| < 1 \) so that \(-er^{1/2}\) is in \( I_r \) and so is \textit{not} in \( E_r \). Therefore, \( H_r \) is harmonic on all of \( E_r \) and so is bounded there by the greatest of its boundary values. Since \( U_r \) vanishes on \( J_r \), one has

\[
H_r(w) \leq M(r) = \sup \{ \ln |G_r(w)| : w \in J_r \}
\]

for all \( w \) in \( E_r \), provided \( 0 < r < 1 \) and \(-e\) is in \( I_r \). [At this point in the proof of (3.1) in \([1]\) the point \(-e\), which there corresponded to the present \(-er^{1/2}\), could not be guaranteed to belong to \( I_r \) for all sufficiently small \( r \); nevertheless this inequality was used as \( r \) approached 0 and error was born.]

If a point \( w \) in \( E_r \) is chosen and if \( r' < r \) then \( w \) is also in \( E_r \) and

\[
H_r(w) = [U_r(w) - U_{r'}(w)] + \ln \left| \frac{w/\left(\frac{r}{r'}\right)^{1/2} + e}{w^{(r')^{1/2}} + e} \right| + H_{r'}(w).
\]

The first bracketed term is not positive because the Green's function is an increasing functional of domain and \( E_r \subset E_{r'} \). The logarithmic term is, for \( r' \) sufficiently close to zero, also not positive. Thus, for \( r' \) close to zero,

\[
H_{r'}(w) \leq M(r').
\]

Because the curves \( J_{r'} \) shrink uniformly to the origin as \( r' \) approaches zero one may conclude that

\[
\lim_{r' \to 0} M(r') = \lim_{r' \to 0} \ln |G_{r'}(w)| = \ln 1 = 0.
\]

Thus \( H_r \leq 0 \) on \( E_r \), as required by (3.1)*.

The statement of Theorem 3.2 as it appears in \([1]\) is (substantially) correct, but is repeated here for the reader's convenience. Its proof is then adapted to the use of (3.2)*.

**Theorem 3.2 (The final inequality).** Let \( D \) be a domain in the sphere with a boundary component \( K \) which contains at least two different points, \( p' \) and \( q \), let \( d \) be a point of \( D \), and let \( T \) be the rational fractional transformation which sends \( d, p, q \) respectively onto \( 0, 1, \infty \), where \( 1 = T(p) \) is the point in \( T(K) \) closest to \( 0 = T(d) \). Then \( D \) admits a Green's function \( U \) with pole at \( d \) such that \( U(t) \to 0 \) as \( t \to p, \ t \in D \). Moreover, if \( H \) is defined by the rule
\[ H(z) = U(T^{-1}(z)) + \ln |z| \]

then \( H \) is harmonic at \( z = T(d) = 0 \) and
\[ H(0) \leq \ln 4. \]

**Proof.** Let \( T \) denote the rational fractional transformation which sends \( d, p, q \) respectively onto \( 0, r, \infty \), where \( 0 < r < 1 \). It is convenient to denote \( T_r(A) \) by \( A(r) \) for any set or point \( A \). In the notation of (3.2)*, the function \( F_r \) sends \( I_r \) onto \( L_r \), the open disk of radius \( r \) and center at the origin. If \( I_r^* \) denotes the reflection in the real axis of \( I_r \) then \( F_r(I_r^*) = L_r \) also. Let \( C_r \) denote the complement in the sphere of \( K(r) \); one checks that \( C_r \) is a simply connected domain containing \( D(r) \) which in turn contains \( L_r \) by choice of \( p \). Since \( C_r \) contains neither \( r \) nor \( \infty \), it follows that \( F_r \) is locally 1-1 on \( F_r^{-1}(C_r) \). Using the simple connectedness of \( C_r \) one may construct (by Lemma 2.4 of [1]) two single-valued branches, \( g_r \) and \( h_r \) of \( F_r^{-1} \). The functions \( g_r \) and \( h_r \) are analytic homeomorphisms defined on \( C_r \) with \( g_r(0) = -ir^{1/2} \) and \( h_r(0) = ir^{1/2} \). Evidently \( g_r \) sends \( L_r \) onto \( I_r^* \) and \( h_r \) sends \( L_r \) onto \( I_r \). Since \( F_r \) is single-valued, this implies that \( g_r(C_r) \) is disjoint from \( I_r = h_r(L_r) \), so that its complement in the sphere contains an open set. By Lemma 2.2 of [1], the domain \( g_r(C_r) \) admits a Green's function so that \( C_r \) and \( D(r) \) do also. Let, therefore, \( u_r \) denote the Green's function for \( D(r) \) with pole at the origin.

With \( e = g_r(0) \), it follows (in the notation of (3.2)*) that \( u_r \leq U_r(g_r) \) on \( D(r) \). Taking inverses, with \( F_r = g_r^{-1} \) on \( g_r(C_r) \), one has
\[ u_r(F_r(w)) \leq U_r(w) \leq \ln \left| \frac{w/(r)^{1/2} + e}{w - e} \right|, \]
according to (3.2)*. If one defines \( H_r \) by the rule
\[ H_r(z) = u_r(z) + \ln |z| \]
then \( H_r \) is harmonic on \( D(r) \) and
\[ H_r(F_r(w)) \leq \ln \left| \frac{w/(r)^{1/2} + e}{w - e} \right| + \ln |F_r(w)|, \]
\[ \leq \ln \left| \frac{w/(r)^{1/2} + e}{w - e} \right| + \ln \left| \frac{F_r(w)}{w - e} \right|. \]

Because \( F_r(0) = 0 \), the ratio \( F_r(w)/(w - e) \) approaches \( F_r(e) = 2e \) as \( w \to e \). It follows that
\[ H_r(F_r(e)) = H_r(0) \leq \ln |-i - i(r)^{1/2}| + \ln |-2i(r)^{1/2}|. \]
Since $u_r = U(T_r^{-1})$ it is clear that $H = H_1$. Since $T_r = rT_1$, it follows that $T_r^{-1}(z) = T_1^{-1}(z/r)$ so that $H_r(z) = H_1(z/r)$. Taking the limit as $r$, $0 < r < 1$, approaches one, one has

$$H(0) = \lim_{r \to 1} H_r(0) \leq \ln 2 + \ln 2 = \ln 4,$$

as required.

**Bibliography**


**Northwestern University**

**SOME DIFFERENTIAL GEOMETRIC PROPERTIES OF SUBMANIFOLDS OF EUCLIDEAN SPACES**

ROBERT E. STONG

1. **Introduction.** The translation theorem of Hopf and Voss [3] has been generalized by Hsiung [4] and Voss [7] to hypersurfaces, by Hsü [6] to other elementary transformations, and these results were obtained for hypersurfaces by Aeppli [1]. In the first sections of this paper, these results will be generalized to $n$-dimensional manifolds in $(n+m)$-dimensional Euclidean space. In the final section, a condition will be given under which a submanifold of Euclidean space is a submanifold of a hypersphere, extending a result of Hsiung [5].

All manifolds mentioned will be assumed to be compact, connected, orientable, $n$-dimensional ($n \geq 2$) manifolds with closed boundaries (empty or of dimension $n - 1$) differentiably imbedded in an $(n+m)$-dimensional Euclidean space $E^{n+m}$ ($m \geq 1$). The notation adopted will be essentially that of Hsiung [4]. The following conventions will be adopted for indices:

$$\alpha, \beta, \cdots = 1, \cdots, n + m,$$

$$\lambda, \mu, \cdots = 1, \cdots, n + m - 1,$$

$$a, b, \cdots = 1, \cdots, m,$$

$$i, j, \cdots = 1, \cdots, n,$$

$$r, s, \cdots = 0, \cdots, n.$$

Considerable use will be made of a vector product like that defined by Hsiung [5]. Namely, if $I_1, \cdots, I_{n+m}$ are a fixed frame of mutually