NOTE ON DEGREES OF PARTIAL FUNCTIONS

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The notion of one function's being recursive in another is normally considered only for full functions; but Davis [1, p. 171] has given a definition applicable also to partial functions. For one-argument functions (to which we restrict ourselves for the sake of simplicity) this reads: \( f \) is partial recursive in \( g \) if there is a completely computable functional \( \Phi \) for which \( f(x) = \Phi(g, x) \). And here \( \Phi \) is called completely computable if for some partial recursive \( h \) we have

\[
\Phi(g, x) = t \iff (\exists y)(y^{(1)} \subseteq g \quad \text{and} \quad h(x, y) = t),
\]

\( \{ i^{(1)} \} \) being an effective enumeration of finite functions. In this note we shall argue that Davis' definition does not do justice to our intuitive idea of relative computability; we shall suggest an alternative definition; and we shall show that on either definition there are degrees which are not degrees of any full function. In fact we shall show that in a suitable sense "almost no" degrees are degrees of full functions.

Let \( \gamma \) be any nonrecursive set, and let \( g_0(2x) = 0 \) for \( x \in \gamma \), \( g_0(2x + 1) = 0 \) for \( x \notin \gamma \), \( g_0(x) \) undefined otherwise. Let \( c_\gamma \) be the characteristic function of \( \gamma \). Then \( c_\gamma(x) \) is 0 if \( g_0(2x) \) is 0 and 1 if \( g_0(2x + 1) \) is 0, so clearly \( c \) is effectively computable from \( g_0 \) in the intuitive sense. None the less, since \( g_0 \) is a restriction of the constant function \( n(x) = 0 \),

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2 We use the notation of [1] except that (1) upper-case Greek letters denote functionals; (2) lower-case Greek letters denote sets (of numbers); (3) upper-case italic letters denote classes (of sets) as well as relations; (4) the complement of a set \( \alpha \) is written \( \alpha' \); (5) sometimes we write \( [\Phi(\alpha)] \) for \( \lambda x(\Phi(f, x) = \alpha) \).

3 This fact was known to Lacombe and Shoenfield as early as 1958; but the use of a category argument to establish it, and the consequent strengthening of the result, is new.

4 We say that almost no semicharacteristic functions (see below) have a certain property, if the class of all sets whose semicharacteristic functions have this property is of first category; and that almost no (weak) degrees have a certain property if almost no semicharacteristic functions belong to (weak) degrees having that property. The latter usage is justified by the fact that every weak degree \( D \) contains a semicharacteristic function (for example \( c_\alpha^g \) where \( \alpha = \{ 2^{x \cdot 3^{(1)}} | f(x) \text{ is defined} \} \) for any \( f \in D \). (Not every strong degree contains a semicharacteristic function; for example this is not true, by the argument of the following paragraph, for any degree of a full non-recursive function.) Observe finally that there exist functions (for example the function \( g_0 \) of the following paragraph) which are not of the same degree as any full function on Davis' definition, though they are on ours.
\( \lambda x \Phi(g_0, x) \) is a restriction of the partial recursive function \( \lambda x \Phi(n, x) \) for every completely computable \( \Phi \): but \( c_n \), being full and nonrecursive, cannot be a restriction of a partial recursive function. Hence \( c \) is not partial recursive in \( g_0 \) in the sense of Davis.

We regard \( f \) as effectively computable from \( g \) in the intuitive sense if there exists a mechanical method by means of which every correct and no incorrect value of \( f \) can be computed using only finitely many values of \( g \). If we assume that this method can be formalized in some formal system with recursive rules of inference, we are led to the following amendment of Davis' definition.

\( f \) is called partial recursive in \( g \) if there is a recursively enumerable relation \( R(x, y, t) \) for which

\[
(1) \quad f(x) = t \leftrightarrow (\exists y)(y \uparrow_{11} \subseteq g \text{ and } R(x, y, t)).
\]

Trivially, if \( f \) is partial recursive in \( g \) in Davis' sense, it is in ours too. For \( g \) full the converse holds (cf. the Corollary of Theorem XIX in [2, p. 331]) but by the immediately preceding counterexample not for \( g \) arbitrary. Two functions are called strongly (Turing) equivalent if each is partial recursive in the other sense of Davis [1, p. 171]; weakly equivalent if each is partial recursive in the other in the sense of our definition. If two functions are strongly equivalent they are weakly equivalent; but not conversely by footnote 4 above. The equivalence classes relative to strong (weak) equivalence are called strong (weak) degrees. Not every strong degree contains a full function (footnote 4). Our main result in this note is that the same is true of weak degrees—in fact (cf. footnote 4) that "almost no" (weak) degree contains a full function. This will be established if we can prove the following theorem.

For each set \( \alpha \), let \( c_\alpha^0 \) (the semicharacteristic function of \( \alpha \)) be that function which is 0 on \( \alpha \) and undefined elsewhere. The class of all sets \( \alpha \) for which some full nonrecursive function is partial recursive in \( c_\alpha^0 \) in the sense of our definition is of first category.\(^6\) A fortiori the same is true of the class of all \( \alpha \) for which some full nonrecursive function is partial recursive in \( c_\alpha^0 \) in Davis' sense, and of the class of all \( \alpha \) for which \( c_\alpha^0 \) is strongly (weakly) equivalent to some full function.

For the proof, call \( f \) partial recursive in \( g \) with Gödel-number \( i \) if (1) holds where \( R \) is the \( i \)-th recursively enumerable relation in some canonical enumeration. If this is so we write \( f = \lceil \Phi_{i g} \rceil \).\(^7\)

\(^6\) We use the topology standard in recursion theory, i.e. we identify sets (or their characteristic functions) with points of \( \{0, 1\}^\mathbb{N} \), where \( \{0, 1\} \) is given the discrete topology. The collection of all classes \( \{\alpha | \beta \subseteq \alpha \text{ and } \alpha \cap \gamma = \emptyset \} \), where \( \beta \) and \( \gamma \) are disjoint finite sets, forms a convenient basis of open classes.
NOTE ON DEGREES OF PARTIAL FUNCTIONS

C. E. SHANKS

1961

521

does not exist for all \( i, g \); in fact it exists if and only if

\[
y_1^{[1]}, y_2^{[1]} \subseteq g, R_i(x, y_1, l_1), R_i(x, y_2, l_2) \rightarrow t_1 = t_2
\]

where \( R_i \) is the \( i \)th recursively enumerable relation.)

The theorem will follow if we can show that the class of all \( \alpha \) for which \( [\Phi_c^0, \alpha] \) is full but not recursive is nowhere dense. Let then \( N \) be any basic open class; we seek a subneighborhood \( N_0 \) of \( N \) such that

\[
\alpha \in N_0, [\Phi_c^0, \alpha] \text{ defined and full } \rightarrow [\Phi_c^0, \alpha] \text{ recursive.}
\]

Let \( N = \{ \alpha | \beta \subseteq \alpha \text{ and } \gamma \cap \alpha = \emptyset \} \). Then \( N_0 \) satisfying (2) is defined by cases as follows.

**Case I.** \( [\Phi_c^0, \gamma] \) is full. Then set \( N_0 = N \). For if \( \alpha \) satisfies the hypothesis of (2) we have \( \alpha \subseteq \gamma', c_\alpha \subseteq c_\gamma, [\Phi_c^0, \alpha] \subseteq [\Phi_c^0, \gamma] \). But since \( [\Phi_c^0, \alpha] \) is full, \( [\Phi_c^0, \alpha] = [\Phi_c^0, \gamma] \) and is therefore recursive.

**Case II.** \( [\Phi_c^0, \gamma] \) is defined, but not full. Again set \( N_0 = N \). For as in Case I, \( \alpha \subseteq N_0 \rightarrow [\Phi_c^0, \alpha] \subseteq [\Phi_c^0, \gamma] \). But then \( [\Phi_c^0, \alpha] \) is not full either, and (2) is vacuously true.

**Case III.** \( [\Phi_c^0, \gamma] \) undefined. This can only happen if

\[
( \exists y_1 y_2 l_1 l_2 x)(y_1^{[1]}, y_2^{[1]} \subseteq c_\gamma, R_i(x, y_1, l_1), R_i(x, y_2, l_2), t_1 \neq t_2).
\]

Let \( \delta \) be the union of the domains of \( y_1^{[1]} \) and \( y_2^{[1]} \) (so that \( \delta \subseteq \gamma' \)). Then we can set \( N_0 = \{ \alpha | \beta \cup \delta \subseteq \alpha \text{ and } \gamma \cap \alpha = \emptyset \} \). For \( \alpha \subseteq N_0 \rightarrow \delta \subseteq \alpha \rightarrow [\Phi_c^0, \alpha] \) undefined; and again (2) holds vacuously.

**Bibliography**


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