IRRATIONAL POWER SERIES

L. J. MORDELL

In a paper with the same name which is to appear in due course in the *Proceedings of the American Mathematical Society*,[1] Dr. Morris Newman proves the

THEOREM. *Let* α *be a real number and let* $f(x)$ *be a polynomial of degree* $\geqq 1$. *Then*

$$F(x) = \sum_{n=0}^{\infty} f([n\alpha])x^n \tag{1}$$

is a rational function of x *if and only if* α *is a rational number.*

Here $[x]$ is the integer part of x. We write $x = [x] + \{x\}$, where $\{x\}$ is the fractional part of x.

He has been good enough to let me see his manuscript and I am much obliged to him for this. His proof is very simple and really of an arithmetical nature. I give here a proof which is perhaps more direct and of an analytical character more appropriate for such questions. His proof makes use of the uniform distribution of $\{n\alpha\}$ where α is irrational, but it suffices for mine that $\{n\alpha\}$ takes an infinity of values for integer values of n.

I prove a slightly more general

THEOREM. *Let* α *be a real number and let* $f(x, y)$ *be a polynomial in* x, y *of degree* $\geqq 1$ *in* y. *Then*

$$F(x) = \sum_{n=0}^{\infty} f(n, \{n\alpha\})x^n, \tag{2}$$

is a rational function of x *if and only if* α *is irrational.*

On replacing $[n\alpha]$ by $n\alpha - \{n\alpha\}$ in (1), it suffices to prove the result for, say,

$$F(x) = \sum_{n=0}^{\infty} f(n, \{n\alpha\})x^n, \tag{3}$$

Hecke in his paper *Ueber analytische Funktionen und die Verteilung von Zahlen* mod *Eins*, Abh. Math. Sem. Univ. Hamburg vol. 1 (1921) pp. 54–76 proves the result when in (1), $f[n\alpha] = [n\alpha]$, in an

Received by the editors August 1, 1960.

[1] Proc. Amer. Math. Soc. vol. 11 (1960) pp. 699–702. (Ed. note.)

entirely different way by using results of Weyl on uniform distribution.

Questions involving polynomials in $\{n\alpha\}$ naturally suggest the Bernoulli polynomials. These are defined by

$$\frac{ze^{zx}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n, \tag{4}$$

and so $B_n(x)$ is a polynomial of degree n in x whose expansion begins with $x^n - (1/2)nx^{n-1} + \cdots$. Also $B_0(x) = 1$, $B_1(x) = x - 1/2$. On multiplying (4) by $e^z - 1$ and equating coefficients of z^{n+1} on both sides, we find

$$x^n = \sum_{r=0}^{n} \frac{n!}{r!(n - r + 1)!} B_r(x). \tag{5}$$

We apply this to the powers of $\{n\alpha\}$ in (3), which then takes the form

$$F(x) = \sum_{n=0}^{\infty}\left(\sum_{r,s=0}^{p} c_{r,s} n^r B_s(\{n\alpha\})\right) x^n, \tag{6}$$

say, where r, s run through all non-negative integer values $\leqq p$ a constant integer independent of n, and the c_{rs} are constants independent of n.

The series (1), (2), (3), (6) converge absolutely for $|x| < 1$. Suppose first that α is rational, say $\alpha = p|q$ where q is a positive integer. Then $\{n\alpha\}$ is a periodic function of n with period q, and obviously $F(x)$ is a rational function of x since $\sum_{n=0}^{\infty} (qn+q_1)^t x^n$ is clearly one when t is a non-negative integer and q_1 is a constant.

Suppose next that α is irrational. We shall show that the circle $|x| = 1$ is a line of essential singularities for $F(x)$ in (6), since the points $x = e^{2m\pi i\alpha}$ for all integers m except a finite number, are singularities of $F(x)$.

We have the well-known Fourier expansion for $s \geqq 0$

$$B_s(\{y\}) = \sum_{m=-\infty}^{\infty} d_{m,s} e^{-2m\pi i y}, \tag{7}$$

where when $s > 0$, $d_{m,s} = 0$ if $m = 0$, $d_{m,s} = (-1)^{s-1} s!/(2m\pi i)^s$ if $m \neq 0$. The series (7) converges absolutely except when $s = 1$, and then $\sum_{-\infty}^{\infty}$ stands for $\lim_{N\to\infty} \sum_{m=-N}^{N}$. The result is easily obtained from (4). Thus

$$d_{m,s} = \int_0^1 B_s(x) e^{2m\pi i x} dx,$$

and so (4) gives

$$\sum_{s=0}^{\infty} \frac{d_{m,s}}{s!} z^s = \int_0^1 \frac{ze^{(2m\pi i+z)x}dx}{e^z - 1} = \frac{z}{2m\pi i + z}.$$

The result follows on equating coefficients of z^s on both sides.

It may simplify the exposition if we consider first the special case when $c_{rs} = 0$ for all $r > 0$ in (6). We then put

$$G(x) = \sum_{n=0}^{\infty} \left(\sum_{s=1}^{p} c_s B_s(\{n\alpha\}) \right) x^n. \tag{8}$$

We deal first with the part of this arising when $s > 1$. We substitute for $B_s(\{n\alpha\})$ from (7) in (8) which becomes an absolutely convergent double series. Sum for n, whence

$$\sum_{n=0}^{\infty} B_s(\{n\alpha\}) x^n = \frac{(-1)^s s!}{(2\pi i)^s} \sum_{m=-\infty}^{m=\infty}{}' \frac{1}{m^s(1 - xe^{-2m\pi i\alpha})}, \tag{9}$$

where the dash denotes the omission of the term with $m = 0$. We considered more generally the series

$$H(x) = \sum_{m=-\infty}^{\infty} \frac{b_m}{1 - xe^{-2m\pi i\alpha}}, \tag{10}$$

where the b's are independent of x and only a finite number of the b's are zero. Suppose that the series $\sum b_m$ converges absolutely. We prove that $|x| = 1$ is a line of essential singularities for $H(x)$, and that $x = e^{-2\lambda\pi i\alpha}$ for all integers λ except a finite number, is a singularity of $H(x)$. On putting $x = e^{-2\lambda\pi i\alpha}y$ and then replacing m by $m - \lambda$, it suffices to prove that if $b_0 = 1$, then $x = 1$ is a singular point of $H(x)$. We show in fact that

$$H(x) \sim \frac{1}{1 - x} \qquad \text{for } 0 < x < 1,\ x \to 1.$$

Write (10) as

$$\begin{aligned} H(x) &= \frac{1}{1-x} + \sum_{m=-N}^{N}{}' \frac{b_m}{1 - xe^{-2m\pi i\alpha}} + \sum_{|m|>N} \frac{b_m}{1 - xe^{-2m\pi i\alpha}} \\ &= \frac{1}{1-x} + H_1(x) + H_2(x) = \frac{1}{1-x} + R(x), \end{aligned}$$

say. Now

$$|1 - xe^{-2m\pi i\alpha}|^2 = 1 - 2x\cos 2m\pi\alpha + x^2 > (1-x)^2$$
$$= (x - \cos 2m\pi\alpha)^2 + \sin^2 2m\pi\alpha \geqq \sin^2 2m\pi\alpha.$$

Take N so great that for given $\epsilon > 0$, $\sum_{|m|>N} |b_m| < \epsilon$. Then

$$|H_2(x)| < \frac{\epsilon}{1-x}, \qquad |H_1(x)| < \sum_{m=-N}^{N} \frac{|b_m|}{|\sin 2n\pi\alpha|} < M,$$

where M is independent of x. Hence

$$|R(x)| < \frac{\epsilon}{1-x} + M,$$

and so for real x with $0 < x < 1$ and $x \to 1$,

$$H(x) \sim \frac{1}{1-x}. \tag{11}$$

Consider next the term with $s = 1$ in (7). Now

$$B_1(\{y\}) = \sum_{m=-\infty}^{\infty}{}' \frac{e^{-2m\pi iy}}{2m\pi i}$$

does not converge absolutely, and so we must proceed rather differently. Write

$$H_0(x) = \sum_{n=0}^{\infty} B_1(\{n\alpha\})x^n$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=-\infty;\, m\neq 0}^{\infty} \frac{e^{-2mn\pi i\alpha}}{2m\pi i}\right)x^n.$$

Let λ be any integer $\neq 0$. Then

$$H_0(xe^{-2\pi i\lambda\alpha}) = \sum_{n=0}^{\infty}\left(\sum_{m=-\infty;\, m\neq 0}^{\infty} \frac{e^{-2(m+\lambda)n\pi i\alpha}}{2m\pi i}\right)x^n$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=-\infty;\, m\neq \lambda}^{\infty} \frac{e^{-2mn\pi i\alpha}}{2(m-\lambda)\pi i}\right)x^n$$

on changing m into $m - \lambda$ which is permissible for the conditionally convergent series. Write

$$G_0(x) = H_0(x) - H_0(xe^{-2\pi i\lambda\alpha})$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=-\infty;\, m\neq 0,\lambda}^{\infty} \frac{-\lambda e^{-2mn\pi i\alpha}}{2m(m-\lambda)\pi i}\right)x^n + \sum_{n=0}^{\infty} \frac{e^{-2\lambda n\pi i\alpha}}{2\lambda\pi i}x^n + \sum_{n=0}^{\infty} \frac{x^n}{2\lambda\pi i}. \tag{12}$$

The two simple series on the right have simple poles at $x=e^{2\lambda\pi i\alpha}$ and $x=1$ respectively. The double series on the right is absolutely convergent, and summing for n, it becomes, say

$$G_1(x) = \sum_{m=-\infty;\, m\neq 0,\lambda}^{\infty} \frac{-\lambda}{2m(m-\lambda)\pi i}\, \frac{1}{1 - xe^{-2m\pi i\alpha}}.$$

Since $\sum 1/m(m-\lambda)$ converges absolutely, the series defines a function of x with $|x|=1$ as a line of essential singularities. Also for $m\neq 0, \lambda$,

$$G_1(x) \sim \frac{-\lambda}{2m(m-\lambda)\pi i}\, \frac{1}{1 - xe^{-2m\pi i\alpha}}$$

for $0<xe^{-2m\pi i\alpha}<1$, and $xe^{-2m\pi i\alpha}\to 1-0$. We have now from (8), for $m\neq 0, \lambda$,

$$(13) \qquad G(x) - G(xe^{-2\lambda\pi i\alpha}) \sim \frac{c}{1 - xe^{-2m\pi i\alpha}},$$

where

$$(14) \qquad c = \sum_{s=1}^{p} \frac{c_s s!(-1)^s}{(2\pi i)^s}\left(\frac{1}{m^s} - \frac{1}{(m-\lambda)^s}\right).$$

Suppose now that for some $m\neq 0, \lambda$, $x=e^{2m\pi i\alpha}$ is not a singular point of the left hand side of (13). Then $c=0$, and then (14) gives a polynomial equation in m and so only a finite number of values of m. Excluding these, $x=e^{2m\pi i\alpha}$ is singular for either $G(x)$ or $G(xe^{-2\lambda\pi i\alpha})$. Hence if there is one m not in this excluded set for which $G(x)$ is not singular, then $G(x)$ is singular for $x=e^{2(\lambda+m)\pi i\alpha}$. Since λ is arbitrary except that $\lambda\neq m$, $G(x)$ has singularities everywhere dense on $|x|=1$.

We consider finally (6) when $r>0$. It is obvious that as $x\to 1$, $\sum_{n=1}^{\infty} n^r x^n \sim k(1-x)^{-r-1}$ where k is a constant. This means that when (6) is transformed as before, the singularity at $x=e^{2m\pi i\alpha}$ is dominated by the term with $r=p$, i.e. by $(1-xe^{-2m\pi i\alpha})^{p+1}$. The argument proceeds exactly as before except that the denominators $1-xe^{-2m\pi i\alpha}$ are replaced by $(1-xe^{-2m\pi i\alpha})^{p+1}$.

This completes the proof.

I should like to thank Professor Davenport for his comments on my manuscript.

ST. JOHN'S COLLEGE, CAMBRIDGE, ENGLAND