AN INEQUALITY FOR CERTAIN PENCILS OF
PLANE CURVES

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1. Introduction. In this paper we use a method similar to one used by Engel in [1] in order to get an inequality relating the degree of the generic curve of a rational pencil of plane curves with the order of its divisor of singularities at a base point of the pencil satisfying some special conditions. In [1] Engel uses a particular case of this relation to give a new proof of the following theorem of Jung:

Consider the affine plane over the field \( k \) of complex numbers. Then any entire Cremona transformation is a product of linear transformations and transformations of the following type:

\[
\begin{align*}
x' &= x, \\
y' &= y + cx^n
\end{align*}
\]

where \( c \in k \) and \( n \) is a positive integer.

For the convenience of the reader we sketch how one can obtain a simple proof of the preceding result using our Corollary A in §7.

Our Lemma B is an abstraction and generalization of a lemma of Jung [2], and is proved in more generality than needed here for further reference.

2. Definitions. Let \( k \) be an algebraically closed field of characteristic zero, \( m \) and \( (m - \alpha) \) two integers, \( d \) their g.c.d., so that \( m = ad \) and \( (m - \alpha) = bd \), where \( a \) and \( b \) are relatively prime.

Consider in the projective plane over \( k \) an irreducible curve of order \( m \) having a singular point \( P \) such that:

1. \( P \) is the center of \( w \) places, all having the same tangent.
2. For each \( i = 1, \ldots, w \) there exists an integer \( t_i \) such that a linear form has order \( a t_i \) or \( b t_i \) at the \( i \)th place.
3. \( \sum t_i = d \) (so that \( \sum a t_i = m, \sum b t_i = m - \alpha \)). Such a point will be called an \( (m, m - \alpha) \)-point. Such a curve will be called an \( (m, m - \alpha) \)-curve. A pencil of curves, depending linearly on a parameter \( \lambda \), whose generic member has an \( (m, m - \alpha) \)-point that is independent of \( \lambda \), and has its tangent independent of \( \lambda \) will be called an \( (m, m - \alpha, \lambda) \)-pencil.

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As the property of being an \((m, m - \alpha)\)-curve is a local one we will use affine coordinates.

3. Lemma A. Given an \((m, m - \alpha, \lambda)\)-pencil, a system of affine coordinates [in the projective plane] can be so chosen that the corresponding equation \(F(x, y, \lambda) = 0\) admits the following factorization:

\[
F(x, y, \lambda) = h \prod_{j \in \mathbb{A}} (y - y_{*j})
\]

where

- \(h\) is a unit in \(\text{Cl} \, k(\lambda)[y][[x^{1/b}]]\),
- the \(y_{*j}\) are nonunits in \(\text{Cl} \, k(\lambda)[[x^{1/b}]]\),
- \(\epsilon\) ranges from 1 to \(n\),
- \(\delta_j\) ranges from 1 to \(b\delta_j\),
- \(t\) is the l.c.m. of the \(t_j\)'s,
- \(\text{Cl}\) denotes algebraic closure and the double bracket formal power-series ring.

Proof. It suffices to take as origin the \((m, m - \alpha)\)-point, and \(y = 0\) as the tangent to the \(n\) corresponding places. The desired factorization is then obtained in the standard way, taking into account the existence of Puiseux-expansions at the origin and the form of the local Galois-group at that point.

Note that for the \(i\)th place there will be \(b\delta_i\) factors in this factorization, each of them of the form:

\[
y - D_0(\theta^{\epsilon i}x^{1/b})^{\delta_i} - \sum_{\tau = 1}^{\infty} D_{\tau}(\theta^{\epsilon i}x^{1/b})^{\delta_i+\tau}
\]

where \(\theta\) is a primitive \(b\delta_i\)th root of unity and \(D_i \in \text{Cl} \, k(\lambda)\).

4. Lemma B. Let \(k\) be any field, \(y, x_1 \cdots x_n\) and \(\lambda\) indeterminates. Suppose we have an identity

\[
(4.1) \quad g(yx_1 \cdots x_n) + \lambda f(yx_1 \cdots x_n) = h \prod_{i = 1}^{m}(y - y_i)
\]

where

- \(g\) and \(f\) are elements of \(k[[x_1 \cdots x_n]][y]\),
- the \(y_i\) are nonunits in \(\text{Cl} \, k(\lambda)[[x_1 \cdots x_n]]\),
- \(h\) is a unit in \(\text{Cl} \, k(\lambda)[[yx_1 \cdots x_n]]\).

Order the terms of the power-series \(y_i\) lexicographically in the \(x_1 \cdots x_m\). In some of the \(y_i\), certain coefficients may depend on \(\lambda\). Suppose that the first such coefficient in every \(y_i\) is separable over \(k(\lambda)\). Then if two at least of the \(y_i\) coincide in their first \(j\) terms the coefficients of these terms do not depend on \(\lambda\).
Proof. Suppose that in \( \sigma \geq 2 \) of the \( m \) power series \( y_\epsilon \), say for \( \epsilon = 1, \ldots, \sigma \), the first term whose coefficient depends on \( \lambda \) is \( a(\lambda)x_1^{t_1} \cdots x_n^{t_n} \), of total degree \( r = \mu_1 + \cdots + \mu_n \), and suppose that these series coincide up to this last term at least.

Define \( y^* \) as a polynomial in the \( x_1 \cdots x_n \) such that
1. \( y^* = y_1 \pmod{\text{terms of degree } r} \),
2. \( y^* \) has as terms of degree \( r \) the same terms as \( y_1 \) up to \( a(\lambda)x_1^{t_1} \cdots x_n^{t_n} \), and instead of this term has the term \( t x_1^{t_1} \cdots x_n^{t_n} \) where \( t \) is an indeterminate and has no higher terms. Note that \( y^* \) is linear in \( t \). In \( Cl k(\lambda) \left[ [y x_1 \cdots x_n t] \right] \) we consider for (4.1) the specialization \( y \rightarrow y^* \). We then get

\[
\begin{align*}
g(y^*x_1 \cdots x_n) + \lambda f(y^*x_1 \cdots x_n) &= h^* \prod_{\epsilon=1}^m (y^* - y_\epsilon) \\
\end{align*}
\]

where \( g(y^*x_1 \cdots x_n) \) and \( f(y^*x_1 \cdots x_n) \) are polynomials in \( t \). Put

\[
\begin{align*}
g(y^*x_1 \cdots x_n) &= g^*(tx_1 \cdots x_n), \\
f(y^*x_1 \cdots x_n) &= f^*(tx_1 \cdots x_n). \\
\end{align*}
\]

Obviously \( h^* \) remains a unit in \( Cl k(\lambda) \left[ [y x_1 \cdots x_n t] \right] \).
In \( (y^* - y_\epsilon) \) the first term of every factor \( (y^* - y_\epsilon) \) will have as coefficient

\[
\begin{align*}
t - a(\lambda) \quad &\text{if } \epsilon = 1, 2, \ldots, \sigma, \\
&\text{and for any other } \epsilon \text{ it may be } t - c_\epsilon(\lambda) \text{ or an expression } d_\epsilon(\lambda) \text{ independent of } t, \text{ or } t. \text{ By hypothesis } a(\lambda), c_\epsilon(\lambda), d_\epsilon(\lambda) \text{ are separable over } k(\lambda). \text{ The first of the lowest degree terms in the right hand side of (4.2) will be obtained by multiplying the first terms in every factor—this gives a term of the form:}
\end{align*}
\]

\[
[t - a(\lambda)]^\sigma \phi(t, \lambda)x_1^{t_1} \cdots x_n^{t_n}
\]

where \( \phi \) is a polynomial in \( t \) whose coefficients are separable over \( k(\lambda) \).
Therefore

\[
g^*(tx_1 \cdots x_n) + \lambda f^*(tx_1 \cdots x_n) = [t - a(\lambda)]^\sigma \phi x_1^{t_1} \cdots x_n^{t_n}
\]

+ different terms of same total degree + terms of higher degree.

Let \( A(t)x_1^{t_1} \cdots x_n^{t_n} \) and \( B(t)x_1^{t_1} \cdots x_n^{t_n} \) be the first of the lowest-degree terms in \( g^*(tx_1 \cdots x_n) \) and \( f^*(tx_1 \cdots x_n) \) respectively—\( A \) and \( B \) are polynomials in \( t \). Then
\[ A(t) + \lambda B(t) = [t - a(\lambda)]^*\phi(t, \lambda). \]

Differentiating with respect to \( \lambda \) we get

\[ B(t) = [t - a(\lambda)]^{*-1}\psi(t, \lambda) \] (\( \psi \) polynomial in \( t \))

and \( B(t) \) has \( t = a(\lambda) \) as a root—a contradiction, since \( B(t) \) is independent of \( \lambda \).

5. Main theorem. Consider an \((m, m-\alpha, \lambda)\)-pencil in whose generic member the order of the divisor of singularities at its \((m, m-\alpha)\)-point is equal to \( c \). Then the following inequality holds:

\[ m^2 + \alpha + n \geq a\alpha + 2m + c. \]

Proof. By Lemma A we can choose a system of coordinates such that the corresponding equation of the \((m, m-\alpha, \lambda)\)-pencil, \( F(x, y, \lambda) = 0 \), admits the factorization:

\[ F = h \prod_{i, i' \neq \epsilon} (y - y_{i\epsilon}). \]

It is well known (see [3]) that the order of the divisor of singularities at \( P \) for a curve \( F = 0 \) is:

\[ c = -\sum_{i=1}^{n} V_{P_i} \left[ \frac{dx}{F_y} \right] \]

the sum ranging over the \( n \) places of \( F = 0 \) at \( P \). For the \((m, m-\alpha)\)-curves we consider we then have:

\[ \sum V_{P_i}[F_y] = c + \sum V_{P_i}[dx] = c + m - \alpha - n. \]

We will now compute an upper bound for \( V_{P_i}[F_y] \) using (5.1).

We have

\[ \sum V_{P_i}[F_y] = \sum V_{P_i} \left[ \frac{\partial}{\partial y} h \prod_{i, i' \neq \epsilon} (y - y_{i\epsilon}) \right]. \]

At \( P \), we identify \( y \) with one of the roots \( y_{i1}, \ldots, y_{id_i} \), say with \( y_{i1} \).

Then

\[ \sum V_{P_i}[F_y] = \sum V_{P_i} \prod_{i, \delta_i \neq 1} (y - y_{i\delta_i}). \]

We compute \( V_{P_i} \prod_{\delta_i \neq 1} (y - y_{i\delta_i}) \) first. Only for \( t_i \) of the \( \delta_i \) will the first coefficient of \( y_{i\delta_i} \) be equal to the first coefficient of \( y_{i1} \), and in that case, by Lemma B, the two expansions \( y_{i\delta_i} \) and \( y_{i1} \) coincide at most up to the first term whose coefficient depends on \( \lambda \). Denote by
\( \rho(\lambda)x(at_i + \beta_i)/bt_i \) the first term in \( y_{i1} \) whose coefficient depends on \( \lambda \). Then
\[
V_{P_i} \prod_{j \neq 1} (y - y_{j1}) \leq (t_i - 1)(at_i + \beta_i) + (bt_i - t_i)at_i.
\]
For the remaining part, \( V_{P_i} \prod_{j \neq 1} (y - y_{j1}) \) a similar argument gives
\[
V_{P_i} \prod_{j \neq 1} (y - y_{j1}) \leq t_i(at_i + \beta_i) + (bt_i - t_i)at_i.
\]
All in all,
\[
V_{P_i} \left[ \frac{\partial}{\partial y} h \prod (y - y_{j1}) \right] \leq (at_i + \beta_i)(\sum t_i - 1) + at_i \left[ \sum_j (bt_j - t_i) \right].
\]
Remembering that \( \sum t_i = d \) we get
\[
V_{P_i}[F_y] \leq abdt_i - at_i + (d - 1)\beta_i.
\]
Taking the sum over all the places centered at \( P \) we have:
\[
(5.6) \sum_i V_{P_i}[F_y] \leq abd^2 + (d - 1)\sum \beta_i - ad.
\]
To evaluate \( \sum \beta_i \) we consider two independent generic members of the pencil, with parameters \( \lambda, \mu \) and the expression:
\[
(5.7) V_{P_i}(y_{i\delta_i}(\mu) - y_{i\delta_i}(\lambda)) \quad \text{with} \quad i\delta_i \neq e\delta_i,
\]
the \( P_i \)'s being taken on the \( \lambda \)-curve.
Using Lemma B we get in this case
\[
V_{P_i}(y_{i\delta_i}(\mu) - y_{i\delta_i}(\lambda)) = V_{P_i}(y_{i\delta_i}(\lambda) - y_{i\delta_i}(\lambda)).
\]
Therefore
\[
(5.8) \sum_i V_{P_i} \prod_{e\delta_i \neq i1} (y_{i1}(\mu) - y_{i\delta_i}(\lambda)) = \sum V_{P_i}[F_y].
\]
But the intersection-multiplicity of two generic members of the pencil at \( P \) is
\[
(5.9) \sum_i V_{P_i} \prod_{e\delta_i \neq i1} (y_{i1}(\mu) - y_{i\delta_i}(\lambda)) = \sum V_{P_i}(y_{i1}(\mu) - y_{i1}(\lambda)).
\]
Moreover \( \sum_i V_{P_i}(y_{i1}(\mu) - y_{i1}(\lambda)) = \sum_i (at_i + \beta_i) = m + \sum \beta_i \). This
multiplicity is \( \leq m^2 \), so that \( \sum V_{P_i}[F_y] + m + \sum \beta_i \leq m^2 \). By (5.2) we have:

\[
\sum \beta_i \leq m^2 - 2m - c + \alpha + n.
\]  

(5.10)

From (5.2) and (5.6) follows:

\[
c + m - \alpha - n \leq abd^2 - ad + (d - 1) \sum \beta_i.
\]

A fortiori,

\[
c + m - \alpha - n \leq abd^2 - ad + (d - 1)(m^2 - 2m - c + \alpha + n)
\]

or finally

\[
dm^2 + ad + nd \geq \alpha m + 2dm + cd.
\]  

(5.11)

6. Proposition. Let the generic curve of an \((m, m-\alpha, \lambda)\)-pencil have as order of its divisor of singularities at its \((m, m-\alpha)\)-point the number \(c\), with \(c > (m-1)(m-2) - 2\). Then \(\alpha\) divides \(m\).

Applying the main theorem,

\[
m^2 + \alpha + n > \alpha a + 2m + (m - 1)(m - 2) - 2 = aa + m^2 - m,
\]

or

\[
m + n > \alpha(a - 1).
\]

Since \(m = ad\) and \(n \leq d\) we have a fortiori

\[
a + 1 > \frac{\alpha}{d} (a - 1).
\]

As both \(a \geq 1\) and \(d | \alpha\) this can only hold for \(\alpha = d\).

Corollary A. If the generic member of an \((m, m-\alpha, \lambda)\)-pencil is a rational curve, and has no other singular point than its \((m-\alpha)\)-point, then \(\alpha\) divides \(m\).

Proof. For the order of the divisor of singularities is \((m-1)(m-2)\) minus twice the genus.

7. Jung’s theorem. Consider the affine plane over an algebraically closed field \(k\) of characteristic 0. Then any automorphism of this plane is a product of linear transformations and transformations of the following type:

\[
x' = x,
\]

\[
y' = y + cx^n,
\]

where \(c \in k\) and \(n\) is a positive integer.
Proof. Let $\sigma$ be an automorphism of the plane given by
\begin{align}
    x' &= f(x, y), \\
    y' &= y(x, y),
\end{align}
where $f$ and $g$ are polynomials, elements of $k[x, y]$ of degree $n$ and $m$ respectively, $n \geq m$. For $n = m = 1$ there is nothing to prove. So we suppose $n \geq 2$. As $\sigma$ is everywhere biregular in the affine plane the jacobian of $f$ and $g$ is constant. It then follows that the highest degree terms of $f$ and $g$, having jacobian zero, are, up to a constant factor, powers of a common polynomial $h$. In the projective plane $\sigma$ defines a birational transformation which is well defined for a generic point of infinity. The same is true for $\sigma^{-1}$, so that to the point at infinity on a generic line of the affine plane there corresponds under $\sigma^{-1}$ only one point at infinity on the corresponding curve. It follows that the polynomial $h$ is a power of a linear form.

By a linear transformation (7.1) becomes:
\begin{align}
    x' &= x^n + f_1(x, y), \\
    y' &= x^m + g_1(x, y)
\end{align}
where $n \geq m$, $n \geq 2$ and the degrees of $f_1$ and $g_1$ are respectively $\leq n - 1$, $\leq m - 1$. After factoring out a linear transformation if necessary we may assume $n > m$.

To the pencil $y = \lambda$, $\lambda$ an indeterminate, there corresponds under $\sigma$ a pencil of rational curves. Consider their parametric representation, obtained from (7.2) by putting $y = \lambda$. It is readily seen that a generic member of this pencil has a unique point at infinity, $P$, independent of $\lambda$, center of only one place, with the line at infinity as tangent. Moreover a line through $P$ intersects the curve in either $n$ or $(n - m)$ points at $P$. Obviously a generic curve of the pencil has no singularities in the affine plane. Such a pencil is a particular case of a $(n, n - m, \lambda)$-pencil, and by Corollary A we get $m \mid n$.

Then $\sigma = w \circ \mu$, where $w$ is the transformation
\begin{align}
    x' &= x'' + y''n/m, \\
    y' &= y'',
\end{align}
and $\mu$ is of the type
\begin{align}
    x'' &= p(x, y), \\
    y'' &= x^m + g_1(x, y),
\end{align}
with $p$ a polynomial of degree less than $n$. This proves the theorem.
ON PROJECTIVE MODULES OVER SEMI-HEREDITARY RINGS

FELIX ALBRECHT

This note contains a proof of the following

Theorem. Each projective module $P$ over a (one-sided) semi-hereditary ring $\Lambda$ is a direct sum of modules, each of which is isomorphic with a finitely generated ideal of $\Lambda$.

This theorem, already known for finitely generated projective modules [1, I, Proposition 6.1], has been recently proved for arbitrary projective modules over commutative semi-hereditary rings by I. Kaplansky [2], who raised the problem of extending it to the non-commutative case.

We recall two results due to Kaplansky:

Any projective module (over an arbitrary ring) is a direct sum of countably generated modules [2, Theorem 1].

If any direct summand $N$ of a countably generated module $M$ is such that each element of $N$ is contained in a finitely generated direct summand, then $M$ is a direct sum of finitely generated modules [2, Lemma 1].

According to these results, it is sufficient to prove the following proposition:

Each element of the module $P$ is contained in a finitely generated direct summand of $P$.

Let $F=P \oplus Q$ be a free module and $x$ be an arbitrary element of $P$. Let $x=\lambda_1 x_1 + \cdots + \lambda_n x_n$ be a representation of the element $x$ in some base for the free module $F$ and let $G$ denote the free submodule

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