A GENERALIZATION OF HIRZEBRUCH POLYNOMIAL AND COBORDISM DECOMPOSITION

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Introduction. In this paper we shall generalize the Hirzebruch polynomial and utilize it for the determination of cobordism components of a compact orientable and differentiable 4k-manifold.

1. Let $X^{4k}$ be a compact orientable and differentiable 4k-manifold. According to Thom's theorem [1] such a manifold is "cobordant" with a polynomial of the complex projective spaces except torsion, i.e.

$$X^{4k} \approx \sum_{i_1 + \cdots + i_t = k} A_{i_1} \cdots P_{2i_1} \cdots P_{2i_t} \text{ mod torsion}$$

where $P_i(c)$ denotes the complex projective space of the complex dimension $i$ and $A_i$'s denotes some rational numbers. The Hirzebruch polynomial is defined as follows [2]

$$\prod_i \frac{(\gamma_i)^{1/2}}{\tgh(\gamma_i)^{1/2}} = \sum_{i=0}^{\infty} L_i(\varpi_1, \cdots, \varpi_i), \quad \sum_i \varpi_i = \prod_i (1 + \gamma_i)$$

where $\varpi_i$ denotes the Pontryagin class of dimension $4i$. It is well known that

$$L_i(\varpi_1, \cdots, \varpi_i)[P_{2i}(c)] = 1,$$

from which we have

$$L_k(\varpi_1, \cdots, \varpi_k)[X^{4k}] = \text{index of } X^{4k} = \sum_{i_1 + \cdots + i_t = k} A_{i_1} \cdots i_t.$$
\[
\prod_i \frac{(\gamma_i)^{1/2}}{\tgh(\gamma_i)^{1/2}} \left( 1 + y \tgh^2 (\gamma_i)^{1/2} \right) = \sum \Gamma_i(y, p_1, \ldots, p_i)
\]
\[
= 1 + \left(y + \frac{1}{3}\right) p_1 + \left\{ p_2 y^2 + \frac{1}{3} (4p_2 - p_1^2) y + \frac{1}{45} (7p_2 - p_1^2) \right\}
\]
\[+ \left\{ p_2 y^2 + \frac{1}{3} (6p_3 - p_1 p_2) y^2 + \frac{1}{15} (17p_3 - 8p_1 p_2 + 2p_2^3) y
\]
\[+ \frac{1}{3^3 \cdot 5 \cdot 7} (62p_3 - 13p_1 p_2 + 2p_2^3) \right\}
\]
\[+ \cdots \]

and
\[
\prod_i \frac{(\gamma_i)^{1/2}}{\tgh(\gamma_i)^{1/2}} \left( 1 + y \tgh^2 (\gamma_i)^{1/2} \right)^{-1} = \sum \Lambda_i(y, p_1, \ldots, p_i)
\]
\[
= 1 + \left(\frac{1}{3} - y\right) p_1
\]
\[+ \left\{ (p_1 - p_2) y^2 + \frac{1}{3} (p_1^2 - 4p_2) y + \frac{1}{45} (7p_2 - p_1^2) \right\}
\]
\[+ \left\{ -(p_3 - 2p_1 p_2 + p_1^3) y^3 + (2p_3 - 3p_1 p_2 - p_1^3) y^2
\]
\[+ \frac{1}{15} (-17p_3 + 8p_1 p_2 - 2p_2^3) y
\]
\[+ \frac{1}{3^3 \cdot 5 \cdot 7} (62p_3 - 13p_1 p_2 + 2p_2^3) \right\}
\]
\[+ \cdots \]

It follows from (1.4) and (1.5) that
\[
(i) \quad \Gamma_i(0, p_1, \ldots, p_i)[X^{4i}] = \text{index of } X^{4i},
\]
\[
(1.7) \quad (ii) \quad \Gamma_i(1, p_1, \ldots, p_i)[X^{4i}] = 2^{4i}(\text{index of } X^{4i}),
\]
\[
(iii) \quad \Gamma_i(-1, p_1, \ldots, p_i)[X^{4i}] = A\text{-genus of } X^{4i} [2, \text{p. 14}].
\]

Moreover we can prove that \(\Gamma_i(y, p_1, \ldots, p_i)[X^{4i}]\) has integral coefficients. The complete proof will be given in another paper but the proof is easy in the case of an almost complex split manifold because \(\Gamma_i(X^{4i})\) decomposes into many virtual indices [2, p. 87].

2. Next we consider the application of our multiplicative series for the determination of cobordism coefficients. We have from (1.1)
\[ \Lambda_k(y, p_1, \cdots, p_k) \ [X^{4k}] \]

(2.1) \[ = \sum_{i_1 + \cdots + i_k = k} A_{i_1} \cdots A_{i_k} [P_{2i_1}(c)] \cdots [P_{2i_k}(c)] \quad (k \leq 4). \]

Comparing the coefficients of \( y^a \)'s \( (a = 0, \cdots, k) \) we have

(2.2) \[ A_2^2 = \frac{1}{5} (-2p_2 + p_1^2) [X^8], \quad A_{11}^2 = \frac{1}{9} (5p_2^2 - 2p_1^4) [X^8], \]

(2.3) \[ A_3^3 = \frac{1}{7} (3p_3 - 3p_1 p_2 + p_1^3) [X^{12}], \]

(2.4) \[ A_2^3 = \frac{1}{15} (-21p_3 + 19p_1 p_2 - 6p_1^3) [X^{12}], \]

\[ A_{111}^3 = \frac{1}{27} (28p_3 - 23p_1 p_2 + 7p_1^3) [X^{12}], \]

\[ A_4^4 = \frac{1}{9} (-4p_4 + 4p_1 p_3 + 2p_2^2 - 4p_1^2 p_2 + p_1^4) [X^{16}], \]

\[ A_{31}^4 = \frac{1}{21} (36p_4 - 33p_1 p_3 - 18p_2^2 + 33p_1^2 p_2 - 8p_1^4) [X^{16}], \]

(3.1) \[ A_{11}^4 = -\frac{1}{45} (180p_4 - 159p_1 p_3 + 80p_2^2 - 150p_1^2 p_2 + 36p_1^4) [X^{16}], \]

\[ A_{111}^4 = -\frac{1}{81} (165p_4 - 137p_1 p_3 - 70p_2^2 + 127p_1^2 p_2 - 30p_1^4) [X^{16}]. \]

3. Next we consider the case where a \( X^{4k} \) is a submanifold of \( X^{4k+2r} \) where we assume that both manifolds be compact orientable and differentiable. Let \( X^{4k} \) be determined by the cohomology classes \( v_1, \cdots, v_r \in H^2(X^{4k+2r}, Z) \). Then we can determine the cobordism coefficients of \( X^{4k} \) by \( v \)'s and the Pontryagin classes of \( X^{4k+2r} \) as follows:

\[ X^8 \subset X^{10}, \]

(3.1) \[ A_2^2 = \frac{1}{5} (-v_5^2 - 2v_1 p_2 + v_1^3 p_2^2) [X^{10}], \]

\[ A_{11}^2 = \frac{1}{9} (3v_5^2 - v_7 p_1 + 5v_1 p_2 - 2v_1^2 p_2) [X^{10}], \]

1 \( T_k \) is also available for this purpose but for \( k \leq 3 \).
$X^{12} \subset X^{14},$

\[ A_3^3 = \frac{1}{7} \{ -v^7 + (3p_3 - 3p_1p_2 + p_1^3)v \} [X^{14}], \]

\[ A_{21} = \frac{1}{15} \{ 8v^7 - v^5p_1 + v^3(2p_2 - p_1^2) + v(-21p_3 + 19p_1p_2 - 6p_1^3) \} [X^{14}], \]

\[ A_{11} = \frac{1}{27} \{ -12v^7 + 3v^5p_1 + (2p_1^2 - 5p_2)v^3 + (28p_3 - 23p_1p_2 + 7p_1^3)v \} [X^{14}], \]

$X^8 \subset X^{12},$

\[ A_2^2 = \left\{ -\frac{1}{5}(v_1v_2 + v_2v_1) + \frac{1}{5}(p_1^2 - 2p_2)v_1v_2 \right\} [X^{12}], \]

\[ A_{11} = \left\{ \frac{1}{3}(v_1v_2 + v_2v_1) + \frac{1}{9}v_1v_2 - \frac{1}{9}(v_1v_2 + v_2v_1)p_1 + \frac{1}{9}(5p_2 - 2p_1^3)v_1v_2 \right\} [X^{12}]. \]

The method used here was as follows [2, p. 87]: From (2.1) replaced by $\Gamma_k$ and the relation

\[ \Gamma_k(y, p_1, \ldots, p_k)[X^{4k}] \]

\[ = \left[ \kappa^{4k+2r} \left\{ \left( \frac{\tgh v_1}{1 + y \tgh^2 v_1} \right) \cdots \left( \frac{\tgh v_r}{1 + y \tgh^2 v_r} \right) \right. \right. \]

\[ \left. \left. \cdots \sum \Gamma_i(y, p_1, \ldots, p_i) \right\} \right] [X^{4k+2r}] \]

we obtain (3.1)-(3.3) by comparing the coefficients of $y^a's.$

**References**


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