1. **Transformation.** Let $\mathbf{R}$ be a $\sigma$-algebra of subsets of $X$, and $\varnothing$ the set of all probability measures $P$ on $\mathbf{R}$. Let $T$ transform $\varnothing$ into itself. For certain sets $E \subseteq \mathbf{R}$, knowledge of $P$ throughout $E$ (i.e., for all subsets of $E$ belonging to $\mathbf{R}$) determines $TP$ throughout $E$. The class of sets having this property will be denoted by $E_T$, or better, since $T$ will be fixed, by $E$. Evidently $E$ contains $0$, $X$, and the complements of atoms. We show that if $E$ is sufficiently large, then $T$ is a linear combination of the identity and a constant. There are applications to the theory of learning and to political theory $[1;3;4;6]$.

**Theorem 1.**

(A) If $E$ contains an algebra $A$ whose Borel extension is $\mathbf{R}$, and if $|\mathbf{R}| > 4$, then $TP = \alpha P + (1 - \alpha) P_0$, where $\alpha \leq 1$ and $P_0 \in \varnothing$.

(B) The converse is true with no restriction on $\mathbf{R}$.

(C) If $\mathbf{R}$ is infinite, then $\alpha \geq 0$.

In the political interpretation, the elements of $X$ are parties (or political positions). $P$ is the distribution of voters, $T$ is the electoral mechanism, and $TP$ the distribution of seats in the legislature. If $T$ is the identity, the mechanism is Proportional Representation. If $T$ is a constant, the political complexion of the legislature is fixed by law. It will be seen from Theorem 3 that $E \subseteq E$ means that if the complement $-E$ unites in a coalition, the effect is independent of whether this occurs before or after the election. $|\mathbf{R}| > 4$ means essentially that there are more than two parties. Part (A) of the theorem is not true for $|\mathbf{R}| = 4$.

In learning theory, $P$ is a probability distribution of responses, and $TP$ is a new distribution resulting from a learning experience. If $T$ is the identity, there is no learning. If $T$ is a constant, this is one-trial learning.

Bush, Mosteller, and Thompson $[4]$ proved an equivalent theorem for the case $\mathbf{R}$ finite and $E = \mathbf{R}$ (Corollary 3 of Theorem 3). Some of their ideas are used in the proof.

Denote by $B$ the class of sets $E$ such that $P(E) = Q(E)$ implies $TP(E) = TQ(E)$, for all $P, Q \in \varnothing$. The importance of $B$ is that for $P(E) = Q(E)$ implies $TP(E) = TQ(E)$, for all $P, Q \in \varnothing$. The importance of $B$ is that for
each proper set $E \subseteq B$, there is a function $\gamma_B$ mapping $[0, 1]$ into itself such that $TP(E) = \gamma_B[P(E)]$. We have also $\gamma_B(0) = 0$.

For any class $S$ of subsets of $X$, let $S^*$ denote the class of sets $E$ for which the complement $-E \subseteq S$.

**Proposition 1.** $E \cap E^* \subseteq B$.

**Proof.** Let $E, -E \subseteq E$. Let $P(E) = Q(E)$ for two members $P, Q$ of $\emptyset$. Define $P' \subseteq P$ as follows. For $A \subseteq R$, let $P'(A) = P(A - E) + A(c)P(E)$, where $c \in E$ and $A(x)$ is the characteristic function of $A$. Define $Q'$ similarly. We have $P' = Q'$ on $E$. Also $P = P'$ and $Q = Q'$ on $-E$. Hence $TP' = TQ'$ on $E$, while $TP = TP'$ and $TQ = TQ'$ on $-E$. In particular, the last two equations are true for $-E$ itself, and, taking complements, also for $E$. We have $TP(E) = TP'(E) = TQ'(E) = TQ(E)$, proving $E \subseteq B$.

Since $A$ is an algebra, $A = A^*$. Thus $B \supseteq E \cap E^* \supseteq A \cap A^* = A$, and so $B \supseteq A$.

Define the set function $u$ on the class $A - \{X\}$ as follows: $u(E) = \gamma_B(0)$. Using the fact that $TP$ is a measure for each $P$, and choosing $P$ so as to vanish on the appropriate sets, it is easy to show that $u$ is a measure on the semiring $A - \{X\}$, and therefore extends uniquely to a measure $u$ on $R \ [5; 7]$. Evidently $u \leq 1$ on $A - \{X\}$, but it would be incorrect to infer that $u(X) \leq 1$.

**Proposition 2.** Let $E \cap F = 0$; $E, F, E \cup F$ proper sets in $B$; $x, y, x + y \in [0, 1]$. Then $\gamma_{E \cup F}(x + y) = \gamma_E(x) + \gamma_F(y)$.

**Proof.** Using all the hypotheses, it is easy to show that there is a probability measure $P$ with $P(E) = x$ and $P(F) = y$. For this $P$,

$$\gamma_{E \cup F}(x + y) = TP(E \cup F) = TP(E) + TP(F) = \gamma_E(x) + \gamma_F(y).$$

If $E$ and $F$ are proper sets in $A$, let $E \sim F$ denote the statement that $\gamma_E(x) - u(E) = \gamma_F(x) - u(F)$ for all $x \in [0, 1]$. $\sim$ is an equivalence relation.

**Proposition 3.** The relation $\sim$ is universal on the proper sets in $A$.

**Proof.** (1) Let $E \subseteq B$, where $\subseteq$ denotes proper inclusion. By Proposition 2, with $F = B - E$ and $y = 0$, $\gamma_B(x) = \gamma_E(x) + \gamma_{B - B}(0)$. Letting $x = 0$, $\gamma_B(0) = \gamma_B(0) + \gamma_{B - B}(0)$. Subtracting, we have $E \sim B$.

(2) If $E \cap F = 0$ and $E \cup F \neq X$, then $E \sim E \cup F \sim F$ by (1).

(3) If $E, F$ are incomparable and $E \cap F \neq 0$, then $E \sim E \cap F \sim F$ by (1).

(4) This leaves only the case $F = -E$. For the first time, we invoke the hypothesis $|R| > 4$, which easily implies $|A| > 4$. Hence $E$ or $F$
must have a proper subset, say $E \supset A$. Then $E \sim A$ by (1) and $A \sim F$ by (2).

In view of Proposition 3 and $|A| > 2$, the equation
\[
\gamma(x) = \gamma_E(x) - u(E)
\]
for $E \subseteq A - \{0, X\}$ defines $\gamma(x)$ uniquely. $\gamma$ maps $[0, 1]$ into $[-1, 1]$.

**Proposition 4.** $\gamma(x) = ax$, with $a \leq 1$.

**Proof.** Let $x, y, x + y \in [0, 1]$. Choose $E, F$ so that $E, F, E \cup F$ are proper sets in $A$, and so that $E \cap F = 0$. Here we have used $|R| > 4$ for the second and last time. By Proposition 2 and the definitions of $u$ and $\gamma$,
\[
\gamma(x + y) + u(E \cup F) = \gamma_{E \cup F}(x + y) = \gamma_E(x) + \gamma_F(y)
\]
\[
= \gamma(x) + u(E) + \gamma(y) + u(F).
\]
Since $u$ is additive, we conclude that $\gamma(x + y) = \gamma(x) + \gamma(y)$. A bounded function of this type is of the stated form. The proof in [2] can be adapted. Obviously, $\alpha \leq 1$.

Thus $TP(E) = \alpha P(E) + u(E)$ for all $E$ in the semiring $A - \{X\}$. If $\alpha \geq 0$, then $\alpha P + u$ is a measure on $R$, equal to $TP$ on $A - \{X\}$, and therefore on $R$. If $\alpha \leq 0$, then $TP - \alpha P$ is a measure on $R$, equal to $u$ on $A - \{X\}$, and therefore on $R$. In either case $TP = \alpha P + u$ on $R$.

In passing, note that
\[
1 = TP(X) = \alpha + u(X).
\]
If $\alpha = 1$, then $TP \equiv P$. If $\alpha < 1$, define $P_\alpha$ by $(1 - \alpha)P_\alpha = u$. The main assertion (A) of Theorem 1 follows.

Assertion (B) is immediate, taking $A = R$. For (C) we require a simple result from set theory. We omit the proof, which is not difficult.

**Lemma.** If $A$ is an infinite algebra of subsets of $X$, then $X$ is the union of a monotone sequence of sets of $A - \{X\}$.

To resume the proof of (C), the infinite cardinality of $R$ implies the same for $A$. Then we have $u(X) = \lim_{n \to \infty} u(E_n)$ for sets $E_n \subseteq A - \{X\}$. But $u \leq 1$ on $A - \{X\}$, and therefore $u(X) \leq 1$. With (1), we have $\alpha \geq 0$.

For applications to special cases, we need the following closure properties of $E$, which are of independent interest.

**Theorem 2.** (A) If $E, F \subseteq E$, and $E \cup F \neq X$, then $E \cap F \subseteq E$.
(B) $E$ is closed with respect to countable union.
Proof. (A) Let \( P \equiv Q \) on \( E \cap F \). Without loss of generality, assume \( P(E) \leq Q(E) \). Define a new probability measure \( P' \) as equal to \( P \) on \( E \) (i.e., throughout \( E \)), equal to \( Q \) on \( F - E \), and arbitrary on \( X - E - F \) except that \( P'(E) + P'(F - E) + P'(X - E - F) = 1 \). For other sets, \( P' \) is defined by additivity. In verification that the values assigned on \( X - E - F \) are feasible, we observe that this set is not empty and that \( P'(E \cup F) \leq Q(E \cup F) \leq 1 \).

Now \( P \equiv P' \) on \( E \) and \( Q \equiv P' \) on \( F \). Hence \( TP \equiv TP' \) on \( E \) and \( TQ \equiv TP' \) on \( F \). The last two identities are true, therefore, on \( E \cap F \). Hence \( TP \equiv TQ \) on \( E \cap F \), and \( E \cap F \subseteq E \).

We remark that when \( E \cup F = X \), (A) is false in the strong sense that given such overlapping incomparable \( E, F \), there exists a \( T \) for which \( E \) and \( F \) are in \( \mathcal{E}_T \), but \( E \cap F \) is not.

(B) Let \( P \equiv Q \) on \( E = \bigcup_{n} E_n \), where \( E_n \subseteq E \). Then \( P \equiv Q \) on \( E_n \), which implies \( TP \equiv TQ \) on \( E_n \) for each \( n \). Let \( \{F_n\} \) be a disjoint sequence having the same partial unions as \( \{E_n\} \). We have \( TP \equiv TQ \) on \( F_n \), since \( F_n \subseteq E_n \). Then \( TP \equiv TQ \) on \( E \) by countable additivity, and \( E \subseteq E \).

The hypothesis of Theorem 1 may be expressed in two parts:

(I) \( |\mathcal{R}| > 4 \), \( E \) contains a class \( \mathcal{S} \) whose Borel extension is \( \mathcal{R} \), and \( X \subseteq \mathcal{S} \).

(II) \( \mathcal{S} \) is a ring.

Proposition 5. In Theorem 1, (II) can be weakened to: \( \mathcal{S} \) is a semiring.

Proof. The class of finite disjoint unions of elements of \( \mathcal{S} \) is a ring [5]. Since it contains \( \mathcal{S} \), this ring generates \( \mathcal{R} \) and contains \( X \). By Theorem 2B, \( E \) contains the ring.

Examples. In all of these, let \( \mathcal{R} \) be the class of Borel sets.

(i) \( X \) = the real line. Let \( \mathcal{E} \) contain all intervals \([\alpha, \beta)\). (Here and in the following it would suffice to take \( \alpha \) and \( \beta \) rational.) Then \( \mathcal{E} \) contains also \([\alpha, \infty)\) and \((-\infty, \alpha)\). With 0 and \( X \), these finite and semi-infinite intervals constitute a semiring. Proposition 5 applies, and \( TP = \alpha P + (1 - \alpha)P_0 \) with \( 0 \leq \alpha \leq 1 \) as in Theorem 1. This is equally true if \( \mathcal{E} \) is assumed instead to contain all proper closed intervals.

(ii) (a) \( X = (0, 1) \), (b) \( X = [0, 1] \), (c) \( X = [0, 1) \). Similar to Example (i).

(iii) \( X \) = Euclidean \( n \)-space (\( n > 1 \)). Let \( \mathcal{E} \) contain all half spaces \( \{x: x_i \geq \alpha\} \) and \( \{x: x_i < \alpha\} \). Then \( \mathcal{E} \) contains all finite intersections of these sets. (This implication is false for \( n = 1 \).) With \( X \) added, these constitute a semiring, Proposition 5 applies, and \( T \) has the form...
stated in Theorem 1. This is true also if $E$ is assumed to contain all slices \( \{ x : x_i \in [\alpha_i, \beta_i) \} \), or alternatively all cells
\[
\{ x : x_i \in [\alpha_i, \beta_i], \ i = 1, \ldots, n \}.
\]

2. Combination. Bush and Mosteller [3] raised the question in learning theory of whether a set $E$ could be shrunk to a point without making $T$ ambiguous on the reduced space. More precisely, let $E \subseteq \mathbb{R}$. A transformation $C$ of $\emptyset$ into itself is called a combination of $E$ if it satisfies

\[
(C1) \quad CP \equiv P \text{ on } -E
\]

and

\[
(C2) \quad P(E) = Q(E) \implies CP \equiv CQ \text{ on } E.
\]

For example, let $c \subseteq E$, and let

\[
(2) \quad CP(A) = P(E - A) + A(c)P(E) \quad \text{for each } A \subseteq \mathbb{R}.
\]

We say that $E \subseteq C$ ($E$ is combinable) if for each combination $C$ of $E$, and for each $P \subseteq \emptyset$, we have

\[
(3) \quad CTCP = CTP.
\]

In learning theory, (3) is called the Combining of Classes condition.

**Theorem 3.** $C = E^*$. 

**Proof.** Let $E \subseteq C$, and $C$ be a combination of $E$. We observe first that (C1) and (C2) imply

\[
(C3) \quad CP = CQ \text{ if and only if } P = Q \text{ on } -E.
\]

Now let $P = Q$ on $-E$. Then $CP = CQ$. Hence $CTP = CTCP = CTQ$. Then a second use of (C3) yields $TP = TQ$ on $-E$. This proves $C \subseteq E^*$. 

Let $-E \subseteq E$. Let $C$ combine $E$. Then $P \equiv CP$ on $-E$, and therefore $TP \equiv TCP$ on $-E$. By (C3), $CTP = CTCP$. Thus $E^* \subseteq C$.

**Corollary 1.** In Theorem 1, the hypothesis that $E$ contains the algebra $A$ can be replaced by $C \supseteq A$.

**Corollary 2.** (A) The union of two overlapping sets of $C$ is in $C$. 
(B) $C$ is closed with respect to countable intersection.

**Corollary 3.** Let $X$ be finite, $|X| > 2$, let $\mathcal{R}$ be the class of all subsets of $X$, and $C = \mathcal{R}$. Then $TP = \alpha P + (1 - \alpha)P_0$.

This is the Bush-Mosteller-Thompson theorem [4] mentioned...
earlier. Bush and Mosteller [3] showed that \( \alpha(|X| - 1) \geq -1 \). This bound is attained.

Regarding Example (ii)(c) as the real numbers modulo 1, let \( \mathbf{C} \) contain all intervals \([\alpha, \beta)\), naturally including the case \( \alpha < 1 < \beta \). With 0, this class is a semiring \( \mathbf{S} \). Since \( \mathbf{S} = \mathbf{S}^* \), also \( \mathbf{E} \supseteq \mathbf{S} \), so that Proposition 5 applies, and \( T \) has the familiar form of Theorem 1. In Example (i) (the real line) the corresponding implication is false, even with the additional assumption that \( \mathbf{C} \) contains all semi-infinite intervals, and similarly for Example (ii). To prove this, we use

**Proposition 6.** Let \( T \) be a combination of \( \mathbf{E} \) of type (2). Then

\[
\mathbf{C}_T = \left( \left\{ \{c\} \right\} \right) + \cup \left( \mathbf{E}_0 \right) - \cup \left( \mathbf{E}_1 \right) \cap \mathbf{R}.
\]

(For any subset \( S \) of \( X \), \( \{S\}^- \) and \( \{S\}^+ \) denote respectively the class of subsets of \( S \) and the class of supersets of \( S \).) The proof is omitted.

Returning to Example (i), let \( T \) be that combination of \( \mathbf{E} = [c, \infty) \) of type (2) which concentrates \( P(E) \) at \( c \). We see that \( \mathbf{C} \) contains all the finite and infinite intervals mentioned above, but that intervals \([\alpha, \beta)\) containing \( c \) are not in \( \mathbf{E} \), and the conclusion of Theorem 1 is false here.

3. **Partition.** A related problem, motivated by learning theory and political theory, is the following. For \( n > 1 \), let \( \varphi_n \) denote the class of all partitions of \( X \) into exactly \( n \) nonempty parts \( X_i \), and let \( \mathfrak{C}_n \) denote the subclass of partitions (called combinable) for which the \( n \)-tuple \([P(X_1), \cdots, P(X_n)]\) uniquely determines \([TP(X_1), \cdots, TP(X_n)]\). It is not difficult to show that if each \( X_i \in \mathbf{C} \), then \((X_1, \cdots, X_n) \in \mathfrak{C}_n \). The converse is false, so that the latter statement is actually weaker than the former. Despite this, we have

**Theorem 4.** If \( \mathfrak{C}_n = \varphi_n \) for some \( n < \log_2 |\mathbf{R}| \), then \( \mathbf{E} = \mathbf{C} = \mathbf{R} \), and \( TP = \alpha P + (1 - \alpha) P_0 \).

**Proof.** First we show that \( E \in \mathbf{E} \) for all \( E \) divisible into \( n - 1 \) (proper) parts. We can assume \( E \neq X \). Let \( P = Q \) on \( E \), and let \( A \subseteq E \). We can express \( A \) as \( \bigcup_i A_i \), where either \( a = n - 1 \) or each \( A_i \) is atomic. If \( a < n - 1 \), then \( E - A = \bigcup_{i+1}^{n-1} A_i \) by the hypothesis on \( E \). In either case, \( P(A_0) = Q(A_0) \) for \( i = 1, \cdots, n - 1 \) and \( P(E) = Q(-E) \). Hence \( TP(A_0) = TQ(A_0) \). Summing from 1 to \( a \), \( TP(A) = TQ(A) \), and \( E \in \mathbf{E} \).

Evidently \( E \in \mathbf{E} \) is proved unless \( E \) consists of the union of fewer than \( n - 1 \) atoms. Let \( E \) be the union of \( n - 2 \) atoms. Since \( |\mathbf{R}| > 2^n \), \(-E \) has three parts, \( A, B, C \). Then \( E \cup A \) and \( E \cup B \) are in \( \mathbf{E} \), their
union is not $X$, and so their intersection $E$ is in $E$ by Theorem 2A. Similarly for $n - 3$, etc. Thus $E = R$, and the remaining statements follow from Theorems 1 and 3.

The theorem is false for $n \geq \log_2 |R|$. Let $\mathcal{S}_n$ denote the class of partitions of $X$ into $n$ nonempty parts $X_i$, each of which is in a fixed semiring $S$ whose Borel extension is $R$. (The notation is suggested by examples where $S$ consists of intervals.)

When $X$ is the real line, and $S$ the class of intervals $[\alpha, \beta)$, $(-\infty, \alpha)$, $[\alpha, \infty)$, the example at the end of §2 shows that $0 \not\equiv \mathcal{S}_n \subseteq \mathcal{C}_n$ for all $n$ does not imply $E = R$. The same is true for $X$ a finite interval. The situation is different for a circle.

Theorem 5. Let $X$ be the set of real numbers modulo 1, and $S$ the class of intervals $[\alpha, \beta)$. If $\mathcal{S}_n \subseteq \mathcal{C}_n$ for some $n$, then $E = C = R$, and $TP = \alpha P + (1 - \alpha) P_0$.

Proof. If $n = 2$, then evidently $S \subseteq B$. With the single exception of Proposition 3, the proof of Theorem 1A applies, with the semiring $S$ replacing the algebra $A$. Proposition 1 is superfluous. We show now that the conclusion of Proposition 3 holds also in the present context. All intervals mentioned are proper, i.e., not 0 or $X$.

(1) Let $I_1 \subseteq I$. If $I - I_1$ is an interval, then $I_1 \sim I$ as in Proposition 3. If $I - I_1$ is not an interval, then it is the union of two disjoint intervals $I_2$ and $I_3$. Moreover, $I_1 \cup I_2$ is an interval. Thus $I_1 \sim I_1 \cup I_2 \sim I$.

(2) If $I_1 \cap I_2 = 0$ and $I_1 \cup I_2 \neq X$, then there is a proper interval $I$ containing $I_1 \cup I_2$. Then $I_1 \sim I \sim I_2$ by (1).

Thus (1) and (2) in the proof of Proposition 3 are true in our present case. (3) and (4) apply unchanged. (This proof that $S \subseteq B$ implies the linearity of $T$ is valid also for $X = \text{the real line with $S$ all $[\alpha, \beta)$, and for $X = \text{Euclidean $n$-space with $S$ all semiclosed cells}$}.) This completes the proof for $n = 2$.

Next, let $n > 2$. Note that $P \equiv Q$ on $I$ if and only if $P \equiv Q$ for all subintervals of $I$ touching an end point. Hence $I \in E$ if and only if $I \in E$ only if the equality of $P$ and $Q$ for all such subintervals implies the same for $TP$ and $TQ$.

Let $P \equiv Q$ on $I$, and let $I_1$ be a subinterval touching an end point. Write the interval $I - I_1$ as the disjoint union $U_3 \cap I_1$, and let $I_2 = - I_1$. (Here we have used the fact that $S = S^*$.) We have $(I_1, \cdots, I_n) \subseteq C_n$, and $P(I_j) = Q(I_j)$ for all $j$. Hence $TP(I_j) = TQ(I_j)$ for all $j$, and in particular for $j = 1$. Since $I_1$ was arbitrary, this proves $I \in E$. Thus $S \subseteq E$.

Using $S = S^*$ again, we have $S \subseteq E \cap E^*$. By Proposition 1, $S \subseteq B$, and the first part of the proof applies.
REFERENCES


