M. Rosenblum in [2] presented a most ingenious proof of the Fuglede and Putnam Theorems by means of entire vector valued functions [1, p. 59]. We will demonstrate that some curious properties of bounded Hilbert space operators can be derived from Rosenblum's argument and similar arguments.

Throughout this text we mean by an "operator" a bounded linear transformation of a Hilbert space into itself. Given an operator \( A \) we mean by "\( \exp A \)" the uniform limit of the series \( I + A + A^2/2! + A^3/3! + A^4/4! + \cdots \). We let \( A^* \) denote the adjoint of the operator \( A \), and let \( z^* \) denote the complex conjugate of the complex number \( z \). A "normal" operator is an operator which commutes with its adjoint.

A critical fact in the Rosenblum proof is that given a normal operator \( A \) and any complex number \( z \), \( \exp (izA) \exp (iz^*A^*) = \exp (izA - iz^*A^*) = \exp (iz^*A^*) \exp (izA) \), and this operator is unitary because \( i(zA + z^*A^*) \) is skew hermitian. Our first result states, among other things, that the converse is true; if the above equations hold for a fixed operator \( A \) and all complex numbers \( z \), then \( A \) is normal.

**Theorem 1.** The following conditions are equivalent for an operator \( A \).

1. \( A^*A = AA^* \),
2. \( \exp (zA) \exp (-zA^*) \) is unitary for all real \( z \),
3. \( \exp (zA) \exp (-z^*A^*) \) and \( \exp (-z^*A^*) \exp (zA) \) are uniformly bounded in complex \( z \).

**Proof.** It is trivial that \((1) \Rightarrow (2), (3)\). Assume \((2)\). Put \( U = \exp (zA) \exp (-zA^*) \), for some real \( z \). \( \exp (-zA) \exp (zA^*) = U^* = U^{-1} = \exp (zA^*) \exp (-zA) \). Thus \( \exp (-zA) \exp (zA^*) = \exp (zA^*) \exp (-zA) \), all real \( z \). Differentiating twice with respect to \( z \) and setting \( z = 0 \), we have \( A^*A = AA^* \). \((2) \Rightarrow (1)\). Now assume \((3)\). For all integers \( n \), \( A \cdot A^n = A^n \cdot A \). For any complex \( z \), \( A \exp (zA) = \exp (zA) A \), \( A = \exp (zA) A \exp (-zA) \). But \( \exp (-z^*A^*) \exp (zA^*) = \exp (-z^*A^*) \exp (zA^*) \exp (-z^*A) \exp (zA) \) is an entire function which, by \((3)\), is bounded in the complex plane. By Liouville's Theorem this operator is constant, and clearly \( \exp (-z^*A^*) A \exp (zA^*) = \exp (0)A \exp (0) = A \). Differentiating \( A \exp (zA^*) = \exp (zA^*) A \) with respect to \( z \) and setting \( z = 0 \) we get \( AA^* = A^*A \). Hence \((3) \Rightarrow (1)\). This completes the proof.

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Observe that if $A$ and $B$ are operators then $AB = BA$ iff $\exp(za) \exp(zb) = \exp(zb) \exp(za)$ for all real $z$. To establish this, differentiate the latter equation twice with respect to $z$ and set $z = 0$.

The next theorem has several significant special cases. When $B = A^*$, (1)–(3) state that $A$ is hermitian (or self adjoint). When $B = -A^*$, (1)–(3) state that $A$ is skew hermitian. When $B = A^*A$, (1)–(3) state that $A$ is a projection; this is due to the fact that if $A = A^*A$, then (taking adjoints) $A^* = A^*A = A$ and (substituting $A$ for $A^*$) $A = A^2$. When $A = T^*T$ and $B = T^*T^*T$, (1)–(3) state that $T$ is a partial isometry. When $A = T^*$ and $B = T^{-1}$, (1)–(3) state that $T$ is unitary.

**Theorem 2.** The following are equivalent for two operators $A$ and $B$.

1. $A = B$,
2. $\exp(za) \exp(z^*b) = \exp(zb) \exp(z^*a)$, all complex $z$,
3. $\exp(za) \exp(-zb) = \exp(za) \exp(-za)$, all real $z$.

**Proof.** Clearly (1) $\Rightarrow$ (2). Assume (2). $\exp(za) \exp(-zb)$ and $\exp(zb) \exp(-za)$ are entire functions which coincide for $z$ purely imaginary, hence for $z$ complex. (2) $\Rightarrow$ (3). Now assume (3). Differentiating (3) with respect to $z$ and setting $z = 0$, we have $A - B = B - A$ and $A = B$. (3) $\Rightarrow$ (1).

Next we find necessary and sufficient conditions that an operator be the zero operator.

**Theorem 3.** For an operator $A$ the following are equivalent.

1. $a = 0$,
2. $\exp(za) = \exp(z^*a^*)$, all complex $z$,
3. $\exp(za) \exp(z^*a^*) = \exp(za^*) \exp(-za)$, all real $z$.

**Proof.** Obviously (1) $\Rightarrow$ (2). Assume (2). For all purely imaginary $z$, $\exp(za) = \exp(-za^*)$. Differentiating with respect to $z$ and setting $z = 0$, we get $A = -A^*$. Then (2) reduces to $\exp(za) = \exp(-za^*)$, and for real $z$, $\exp(2za) = I$. Differentiating with respect to $z$ and setting $z = 0$ again we get $A = 0$. (2) $\Rightarrow$ (1). Now assume (3). Differentiating (3) twice and setting $z = 0$ again we get $A^*A = -AA^*$ and $A^*A + AA^* = 0$. Hence $A^*A = 0$ and $A = 0$. (3) $\Rightarrow$ (1).

**References**


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