HOLONOMY, RICCI TENSOR AND KILLING VECTOR FIELDS

KATSUMI NOMIZU

1. Let $M$ be a Riemannian manifold, connected and of class $C^\infty$. For any vector field $X$ on $M$, we define a tensor field $A_X$ of type $(1, 1)$, namely, a field of linear endomorphisms of the tangent space at each point, by setting $A_X Y = - \nabla_Y X$, where $Y$ is a tangent vector at an arbitrary point and $\nabla_Y$ denotes covariant derivative with respect to $Y$.

It is known [2] that if $A$ is a Killing vector field, then $A_X$ is a skew-symmetric endomorphism of the tangent space and belongs to the normalizer of the holonomy algebra (Lie algebra of the homogeneous holonomy group) at each point of $M$. A result of Lichnerowicz [3] implies that if the restricted homogeneous holonomy group is irreducible and if the Ricci tensor is not zero, then $A_X$ belongs to the holonomy algebra at each point. One of the basic contributions in contemporary Riemannian geometry is the result, due to Kostant [2], that the same conclusion holds if $M$ is compact. His proof uses the Green-Stokes formula which is valid only on a compact Riemannian manifold.

In the present note, we wish to provide a more geometric proof to this theorem of Kostant, in fact, in a generalized form where the compactness of the space is not assumed. Namely, we shall prove

**Theorem.** Let $M$ be a complete Riemannian manifold. If $X$ is a Killing vector field defined on $M$ which attains a local maximum in its length at some point of $M$, then $A_X$ belongs to the holonomy algebra at each point of $M$.

Here we say that the length $\|X\|$ of a vector field $X$ attains a local maximum at a point $x \in M$ if $x$ has a neighborhood $U$ such that $\|X\|_y \leq \|X\|_x$ for every point $y \in U$. The assumption of our theorem is valid, for example, if $X$ has constant length on $M$, or if $M$ is compact. The theorem of Kostant follows immediately.

2. We now sketch the proof of our theorem. First, we may assume that $M$ is simply connected. Otherwise, let $\hat{M}$ be the universal covering manifold of $M$ provided with a natural Riemannian metric so
that the projection $\pi$ of $\hat{M}$ onto $M$ is a local isometry. $\hat{M}$ is complete. The holonomy algebra at $x \in \hat{M}$ is the same as the holonomy algebra at $x = \pi(x)$. The given Killing vector field $X$ on $M$ can be lifted to a Killing vector field $\tilde{X}$ on $\hat{M}$ which projects upon $X$. Suppose that $||X||$ attains a local maximum at $x \in M$. It is then clear that $||\tilde{X}||$ attains a local maximum at any point $\tilde{x} \in \hat{M}$ such that $\pi(\tilde{x}) = x$. If the conclusion of the theorem holds for $A_{\tilde{x}}$, then it holds obviously for $A_x$.

Thus, let us assume that $M$ is simply connected and complete. Let $M = M_0 \times M_1 \times \cdots \times M_k$ be the de Rham decomposition of $M$, where $M_0$ is a euclidean space and each $M_i, 1 \leq i \leq k$, is irreducible [4]. The given Killing vector field $X$ on $M$ decomposes into a sum $X_0 + X_1 + \cdots + X_k$, where each $X_i, 0 \leq i \leq k$, is a Killing vector field on $M_i$. Let $x = (x_0, x_1, \cdots, x_k) \in M_0 \times M_1 \times \cdots \times M_k$ be a point where $||X||$ attains a local maximum. Since $||X||^2 = ||X_0||^2 + \cdots + ||X_k||^2$, each vector field $X_i$ on $M_i$ attains a local maximum at the point $x_i$ for the following reason. Suppose that this is not the case and that for some $i, x_i \in M_i$ has an arbitrarily nearby point $y_i \in M_i$ such that $||X_i||_{y_i} > ||X_i||_{x_i}$. Then we can get a point $y = (x_0, x_1, \cdots, y_i, \cdots, x_k)$ at which $||X||_y > ||X||_x$ and which is arbitrarily near $x$, contrary to the assumption that $||X||$ attains a local maximum at $x$. It is clear that we have only to prove the theorem for each $X_i$ on $M_i$.

We now consider each vector field $X_i$ on $M_i$. On the euclidean space $M_0$, the length of a Killing vector field $X_0$ cannot have a local maximum unless it is constant, in which case, $X_0$ is a parallel vector field and the corresponding endomorphism $A_{X_0}$ is zero at every point. On each $M_i, 1 \leq i \leq k$, we make the following argument. The holonomy algebra of $M_i$ is irreducible. If the Ricci tensor of $M_i$ is not identically zero, then $A_{X_i}$ belongs to the holonomy algebra by the result of Lichnerowicz as we already mentioned. To deal with the factor $M_i$ whose Ricci tensor is identically zero, we need the following two general lemmas whose proofs will be given in §4.

**Lemma 1.** Let $X$ be a Killing vector field on a Riemannian manifold. Then

$$\text{div}(A_X \cdot X) = -S(X, X) - \text{trace}(A_X^2),$$

where $S(X, X)$ is the quadratic form in $X$ given by the Ricci tensor.

**Lemma 2.** For a Killing vector field $X$ on a Riemannian manifold with Riemannian metric $g$, let $\phi = (1/2)||X||^2$. For any vector field $V$ with $\nabla V = 0$ in a neighborhood of a point $x$, we have
\[ V^2 \phi = g(V, \nabla_V(A_X \cdot X)). \]

Now assume that the Ricci tensor \( S \) is identically zero. Lemma 1 gives \( \text{div}(A_X \cdot X) = -\text{trace}(A_X)^2 \). If \( \|X\| \) attains a local maximum at \( x \), we have \( V^2 \phi = g(V, \nabla_V(A_X \cdot X)) \leq 0 \) at \( x \) in Lemma 2. Since \( \text{div}(A_X \cdot X) \) is the trace of the linear mapping \( V \mapsto \nabla_V(A_X \cdot X) \) of the tangent space at \( x \) into itself, we see that \( \text{div}(A_X \cdot X) \leq 0 \) at \( x \). On the other hand, \( A_X \) being skew-symmetric, we have \( \text{trace}(A_X)^2 \leq 0 \) at \( x \). Therefore we must have \( \text{div}(A_X \cdot X) = \text{trace}(A_X)^2 = 0 \) at \( x \), which is possible only when \( A_X = 0 \) at \( x \). Thus \( A_X \) belongs, of course, to the holonomy algebra at \( x \). As was shown in [2], it follows that \( A_X \) belongs to the holonomy algebra at each point of \( M \). This concludes the proof of our theorem.

3. A similar argument allows us to prove a theorem of Bochner [1] in the following form.

**Theorem.** Let \( M \) be a Riemannian manifold whose Ricci tensor is negative definite. If a Killing vector field \( X \) attains a local maximum in its length at some point of \( M \), then \( X \) is identically zero.

In fact, by the same argument following the above lemmas, we have \( -S(X, X) - \text{trace}(A_X)^2 \leq 0 \) at \( x \). On the other hand, since \( S \) is negative definite, we must have \( -S(X, X) \geq 0 \) everywhere. We have also \( -\text{trace}(A_X)^2 \geq 0 \). Thus, at \( x \), we have \( S(X, X) = 0 \) and \( \text{trace}(A_X)^2 = 0 \), which imply that \( X = 0 \) and \( A_X = 0 \) at \( x \). By a well known fact that a Killing vector field on a connected Riemannian manifold is uniquely determined by the values of \( X \) and \( A_X \) at an arbitrary single point [2], we see that \( X \) is zero on the whole manifold.

4. For the sake of completeness, we shall give here proofs of Lemmas 1 and 2.

**Proof of Lemma 1.** The Ricci tensor is given, by definition, by \( S(X, Y) = \text{trace} \text{ of the linear endomorphism } V \mapsto R(V, X)Y \text{ of the tangent space at each point, where } R \text{ is the curvature tensor and } R(V, Y) \text{ is the skew-symmetric endomorphism obtained by contraction of } R \text{ with vectors } V \text{ and } Y. \text{Now assume that } X \text{ is a Killing vector field and } Y \text{ is an arbitrary vector field. We have } \nabla_V(A_X) = R(X, V) \text{ (see [2]), and hence } -R(X, V)Y = -((\nabla_V(A_X))Y = -\nabla_V(A_X \cdot Y) + A_X(\nabla_V Y) = -\nabla_V(A_X \cdot Y) - A_X \cdot A_Y \cdot V. \text{ Thus we obtain } S(X, Y) = -\text{div}(A_X \cdot Y) - \text{trace}(A_X A_Y). \)

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\* After the completion of this paper, there appeared a paper by Robert Hermann, *Totally geodesic orbits of groups of isometries* (Lincoln Laboratory, MIT, June 1960). He makes use of formulas which are essentially the same as ours.
The formula in Lemma 1 follows by taking $Y = X$.

**Proof of Lemma 2.** We recall that $Z \cdot g(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$ for arbitrary vector fields $X$, $Y$ and $Z$ (this is an infinitesimal expression of the fact that the parallel displacement of the Riemannian connection is isometric). Applying this formula, we have

$$V \cdot \phi = \frac{1}{2} V \cdot g(X, X) = g(\nabla_V X, X) = g(-A_X \cdot V, X) = g(V, A_X \cdot X),$$

since $A_X$ is skew-symmetric when $X$ is a Killing vector field. We then obtain

$$V^2 \phi = V \cdot g(V, A_X X) = g(\nabla_V V, A_X \cdot X) + g(V, \nabla_V (A_X)) = g(V, \nabla_V (A_X)),$$

since $\nabla_V V = 0$ by assumption.

**Bibliography**


**Brown University and Catholic University of America**