A PROPERTY OF REGULAR MEASURES IN LOCALLY COMPACT HAUSDORFF SPACES

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1. Introduction. Let $G$ denote a locally compact Hausdorff topological space. $\mathcal{B}$ is the class of Borel sets of $G$, i.e., the $\sigma$-ring generated by the class $\mathcal{C}$ of compact subsets of $G$. Let $\mathcal{D}$ be the class of sets whose intersection with every compact set is a Borel set. Note that $\mathcal{D}$ is a $\sigma$-field containing the open sets of $G$. Given a measure $\mu$ (a non-negative countably additive set function not identically equal to zero) on $\mathcal{B}$, denote by $\mathcal{D}_\mu$ the class of sets which either belong to $\mathcal{B}$ or which differ from a member of $\mathcal{B}$ by a subset of a set of $\mu$-measure zero.

A measure $\mu$ defined on $\mathcal{B}$ is said to be regular if (i) $\mu(C) < \infty$ for every $C \in \mathcal{C}$ and (ii) for every $B \in \mathcal{B}$, $\mu(B) = \sup \{\mu(C) : C \in \mathcal{C}, C \subseteq B\}$. Given a regular measure $\mu$ on $\mathcal{B}$ extend it to $\mathcal{D}$ by defining $\mu(D) = \sup \{\mu(D \cap C) : C \text{ compact} \}$ for all $D \in \mathcal{D}$. Observe that $\mu$ so extended to $\mathcal{D}$ is regular. Below we shall assume that either $\mu$ on $\mathcal{B}$ is given or that $\mu$ on $\mathcal{D}$ is extended to $\mathcal{D}$ as above.

The object of this paper is to prove the following

**Theorem.** To every regular measure $\mu$ on $\mathcal{D}$ there corresponds a unique closed set $A_\mu$, the carrier of $\mu$, with $\mu(G \sim A_\mu) = 0$ and $\mu(U) > 0$ for every non-null relatively open subset $U$ of $A_\mu$.

This theorem when $G$ is compact is due to Wendel [1]. His proof is direct using heavily compactness of $G$.

2. Proof of Theorem. As $\mu \neq 0$ and as $\mu$ is regular we can find a compact set $C$ with $\mu(C) > 0$. As the space is locally compact and as $C$ is a compact subset we can, by considering the covering of $C$ by compact neighbourhoods of the points of $C$, find an open set $A \supseteq C$ with $\overline{A}$ compact. It is clear that $\mu(A) > 0$ since $A \supseteq C$. A slight modification of the proof of [1, Lemma 3] then yields a compact set $A_1 \subseteq \overline{A}$ such that every non-null relatively open subset of $A_1$ has positive $\mu$-measure and $\mu(\overline{A} \sim A_1) = 0$. Write $B = A \cap A_1$. As $A \in \mathcal{B}$ and $A_1 \in \mathcal{B}$ it follows that $B \in \mathcal{B}$. Also $\mu(A \sim B) = 0$ and every non-null relatively open subset of $B$ has positive $\mu$-measure.

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Consider now the collection of all open subsets $A$ which admit of a subset $B \in \mathcal{D}_\mu$ whose non-null relatively open subsets have positive measure and $\mu(A \sim B) = 0$. Let $\mathcal{E}$ denote the family of all such pairs $(A, B)$. In view of earlier remarks this class is nonempty. Partially order this class by: $(A_1, B_1) \leq (A_2, B_2)$ if and only if $A_1 \subseteq A_2$ and $B_1 = A_2 \setminus B_2$.

We will show that every chain in $\mathcal{E}$ has an upper bound. It would then follow that $\mathcal{E}$ has a maximal element.

Let $\{(A_\alpha, B_\alpha), \alpha \in \Sigma\}$ be a chain from $\mathcal{E}$. Write $A = \bigcup_{\alpha \in \Sigma} A_\alpha$; $B = \bigcup_{\alpha \in \Sigma} B_\alpha$. $A$ is thus an open set and hence belongs to $\mathcal{D}_\mu$. We will show that $B \in \mathcal{D}_\mu$ and that non-null relatively open subsets of $B$ have positive measure.

Let $\mu^*, \mu_*$ be respectively the outer and inner measures induced by $\mu$ and let $K$ be any compact subset. Suppose that $C \subseteq A \cap K$, $C$ a compact subset. Hence $C \subseteq A$ and $\{A_\alpha, \alpha \in \Sigma\}$ is an open covering for the compact set $C$. Therefore there exists a finite subcovering from this. As the $A_\alpha$’s are linearly ordered, it follows that $C \subseteq A_\theta$ for some $\theta \in \Sigma$. Now, $\mu(C) = \mu(C \cap A_\theta) = \mu(C \cap B_\theta)$ since $\mu(A_\theta \sim B_\theta) = 0$. Write $C_1 = C \cap B_\theta$. Hence $C_1$ is a Borel set, $\mu(C) = \mu(C_1)$ and $C_1 \subseteq B \cap K$. Regularity of $\mu$ now gives $\mu_*(B \cap K) \geq \mu(A \cap K)$. That $\mu^*(B \cap K) \leq \mu(A \cap K)$ follows from the fact that $B \cap K \subseteq A \cap K$. Hence

$$\mu^*(B \cap K) = \mu_*(B \cap K) = \mu(A \cap K).$$

This implies that $B \cap K \in \mathcal{D}_\mu$. As the compact set $K$ is arbitrary, $B \in \mathcal{D}_\mu$. Also $\mu(B \cap K) = \mu(A \cap K)$. As $\mu$ is regular, we have $\mu(A \sim B) = 0$.

Let $V$ be any open set with $B \cap V$ non-null. Hence there is a $B_\alpha$, $\alpha \in \Sigma$ such that $B_\alpha \cap V$ is non-null. Therefore

$$\mu(B \cap V) \geq \mu(B_\alpha \cap V) > 0.$$ 

Thus $(A, B) \in \mathcal{E}$. That $(A_\alpha, B_\alpha) \leq (A, B)$ for all $\alpha \in \Sigma$ is evident. We have therefore proved that every chain from $\mathcal{E}$ has an upper bound. This implies that $\mathcal{E}$ has a maximal element. Let this be $(M, N)$.

We claim $M = G$. For, if not, two cases can arise.

Case (i) $\mu(G \sim M) = 0$. In this case we see immediately that $(G, N) \in \mathcal{E}$. Further $(M, N) \leq (G, N)$ and the two elements are not the same, thus contradicting the maximality of $(M, N)$.

Case (ii) $\mu(G \sim M) > 0$. By the regularity of $\mu$, there exists therefore a compact set $K \subseteq (G \sim M)$ with $\mu(K) > 0$. As explained at the beginning of the proof, we can find an open set $U \supseteq K$ such that $\overline{U}$ is compact. From all this we conclude that there exists a point $x \in (G \sim M)$
and an open set $U$ such that $x \in U$, $\mathcal{U}$ is compact and $\mu(U) > 0$. We can then find a Borel set $V$ such that $(U, V) \in \mathcal{E}$. Let $M_1 = M \cup U$ and $N_1 = N \cup V_1$ where $V_1 = (V \sim M)$. From the fact that non-null relatively open subsets of $N$ and of $V$ have positive measures we see that $N_1$ has this property too. Notice also that $M_1$ is open, $N_1 \subseteq M_1$ and $\mu(M_1 \sim N_1) = 0$. Thus $(M_1, N_1) \in \mathcal{E}$. Obviously however $M \subseteq M_1$ and $N = M \cap N_1$. So $(M, N) \leq (M_1, N_1)$ and these two elements of $\mathcal{E}$ are not the same. This contradicts the maximality of $(M, N)$. Therefore $G = M$, as we claimed.

Let $A_\mu = \overline{N}$. Then $\mu(G \sim A_\mu) = 0$.

Every open set having a non-null intersection with $\overline{N}$ has a non-null intersection with $N$. Therefore every non-null relatively open subset of $A_\mu$ has positive measure.

To prove the uniqueness of $A_\mu$, assume, if possible, a second closed set $A \neq A_\mu$ such that $\mu(G \sim A) = 0$ and such that every non-null relatively open subset of $A$ has positive $\mu$-measure. If $A \not\subseteq A_\mu$, then $A \sim A_\mu$ is a non-null relatively open subset of $A$. Therefore $0 < \mu(A \sim A_\mu) \leq \mu(G \sim A_\mu) = 0$, a contradiction. Hence $A \subseteq A_\mu$. Similarly $A_\mu \subseteq A$. Thus $A = A_\mu$ and the proof is complete.

Reference


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