A NOTE ON MATRIX RICCATI SYSTEMS

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1. Let \( A(t) \) be an \( n \times n \) matrix, continuous on an interval \( I \); let \( n_1, \ldots, n_k \) be positive integers such that \( \sum_{i=1}^{k} n_i = n \). Let \( A \) be partitioned into submatrices \( A_{ij} \) which are \( n_i \times n_j \), \( (i, j = 1, \ldots, k) \); let \( E_m \) be the identity matrix of order \( m \); and let \( A_m = (A_{m1} \cdots A_{mk}) \). In this note we consider the matrix Riccati system with side condition

\[
Y' = -YA_mY + AY, \quad Y_m(t_0) = E_m,
\]

where \( Y = \text{col}(Y_1, \ldots, Y_k) \) and \( Y_i \) is \( n_i \times n_m \). This equation is derived in a natural way as a generalization of the so-called Riccati system \([1; 2]\). Mainly we generalize some results of Levin \([3]\), who treats the equation

\[
\Gamma' = -\Gamma G_3 \Gamma - \Gamma G_4 + G_1 \Gamma + G_2,
\]

where \( G_1, G_2, G_3, \) and \( G_4 \) are \( n_1 \times n_1, n_1 \times n_2, n_2 \times n_1, \) and \( n_2 \times n_2 \) respectively, and \( \Gamma \) is \( n_1 \times n_2 \). To see that (2) is a case of (1), take \( m = 2 \) and

\[
A = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}, \quad Y(t) = \begin{pmatrix} \Gamma(t) \\ E_n \end{pmatrix}.
\]

For other related results see \([4]\).

2. Let \( X \) be \( n \times n \) and such that

\[
X' = AX.
\]

The partitioning of \( A \) induces a partitioning of \( X \) into submatrices \( X_{ij} \) \((i, j = 1, \ldots, k)\). Let \( X_m = \text{col}(X_{1m} \cdots X_{km}) \). Then

\[
X_m' = AX_m,
\]

and, at least formally,

\[
(X_{mm}^{-1})' = -X_{mm}^{-1}A_mX_{mm}^{-1}.
\]

Thus, if \( X_{mm}^{-1} \) exists on some interval \( I_0 \subseteq I \), \( X_mX_{mm}^{-1} \) is a solution on \( I_0 \) of (1). Further, if \( Y \) is a solution of the differential equation of (1), \( Y_m \) satisfies an equation of the form \( P' + HP = H \); thus from classical

Presented to the Society, January 25, 1961 under the title The cross-ratio property for the matrix Riccati equation; received by the editors June 13, 1960 and, in revised form, September 12, 1960.

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existence and uniqueness theorems (1) has a local existence and uniqueness theorem and $Y_m(t) = E_m$. Gathering together these remarks, we have

**Theorem 1.** Let $X(t)$ be the solution of (3) such that $X(t_0) = E_n$. Then the general solution near $t_0$ of (1) is $X(t)C(\sum_{i=1}^k X_i C_i)^{-1}$, where $C$ is constant and arbitrary, $C = \text{col}(C_1 \cdot \cdot \cdot C_k)$, $C_i$ is $n_i \times n_m$, and $C_m$ is nonsingular.

3. Now let $n = rk$ and $n_i = r$ ($i = 1, \cdot \cdot \cdot , k$). We proceed to write the general solution of (1) in terms of solutions of (1) rather than solutions of (3). Let $U_1, \cdot \cdot \cdot , U_k$ denote solutions of (1); let $U = (U_1 \cdot \cdot \cdot U_k)$; let $Z = \text{diag}(Z_j)$, where $Z_j$ is $r \times r$, and 

$$Z_j = A_m U_j Z_j \quad (j = 1, \cdot \cdot \cdot , k).$$

**Theorem 2.** Let $U(t_0)$ and $Z(t_0)$ be nonsingular; let $C = \text{col}(C_j)$, the $C_j$ being $r \times r$, constant, and arbitrary except that $\sum_{j=1}^k Z_j(t_0) C_j$ is nonsingular. Then the general solution of (1) can be written near $t_0$ as

$$Y = UZC \cdot \left( \sum_{j=1}^k U_m Z_j C_j \right)^{-1}. \quad (7)$$

**Proof.** It is easily verified that $UZ$ satisfies (3) and that, if $U(t_0)$ is nonsingular, $U(t)$ is nonsingular where it exists; the conclusion then follows.

Now let $U_{hij} = U_{hi} - U_{hj}$ and $V_{hij} = U_{hij}^{-1}$. It is easily verified that $(U_i - U_j) V_{hij}$ satisfies an equation of the form (1). Thus, if

$$[(U_i - U_q) V_{hij} - (U_i - U_j) V_{hij}]_{t=t_0} = 0, \quad (8)$$

it is identically zero near $t_0$. Hence, if (8) is satisfied, $V_{hij} U_{hij}$ satisfies

$$W' = A_m U_j W - W A_m U_q. \quad (9)$$

Now fix $q$; we may as well take $q = 1$.

**Theorem 3.** Let $|U(t_0)| \neq 0$. Let $i = k + j - 1$ ($j = 2, \cdot \cdot \cdot , k$). Let $U_i$ be a solution of (1) such that, for some $r_j$ ($1 \leq r_j \leq k$, $r_j \neq m$), $U_{r_ij}$ and $U_{r_1}$ are nonsingular at $t_0$ and (8) is satisfied for $q = 1$ and $h = r_j$. Let $|Z(t_0)| \neq 0$. Then the general solution of (1) can be written in the form (7), where $Z_j = V_{r_ij} U_{r_1} Z_j$ ($j = 2, \cdot \cdot \cdot , k$).

**Proof.** Note that $Z_j = A_m U_1 Z_j$ and that $V_{r_ij} U_{r_1}$ satisfies (9) with $q = 1$; then apply Theorem 2.

We remark that, given $q$, $j$, and $h \neq k$, there does exist a set of initial values for the $U_i$ which satisfies (8). However, if for $U_j$ and
$U_p \ (p \neq j)$ the same initial values for $U_i$ satisfy (8), there is a linear combination of columns of $U_i$, $U_j$, and $U_p$ which is zero. Thus, since $|U(t_0)| \neq 0$, each $U_j$ requires a distinct $U_i$. Thus, although we need not eliminate all of the $Z_j$'s ($j = 2, \ldots, k$) in the general solution, we do need a distinct $U_i$ for each one we do eliminate.

For $k = 2$ we have Theorem 5 of [3]. For $r = 1$ and general $k$ we have a form for $Y$ different from that given in [2].

**BIBLIOGRAPHY**