A NOTE ON MATRIX RICCATI SYSTEMS

W. J. COLES

1. Let $A(t)$ be an $n\times n$ matrix, continuous on an interval I; let $n_1, \cdots, n_k$ be positive integers such that $\sum_{i=1}^{k} n_i = n$. Let A be partitioned into submatrices A_{ij} which are $n_i\times n_j$, $(i, j=1, \cdots, k)$; let E_m be the identity matrix of order m; and let $A_m=(A_{m1} \cdots A_{mk})$. In this note we consider the matrix Riccati system with side condition

$$Y' = -YA_mY + AY, \qquad Y_m(t_0) = E_{n_m}, \tag{1}$$

where $Y=\mathrm{col}(Y_1, \cdots, Y_k)$ and Y_i is $n_i\times n_m$. This equation is derived in a natural way as a generalization of the so-called Riccati system [1; 2]. Mainly we generalize some results of Levin [3], who treats the equation

$$\Gamma' = -\Gamma G_3\Gamma - \Gamma G_4 + G_1\Gamma + G_2, \tag{2}$$

where G_1, G_2, G_3, and G_4 are $n_1\times n_1$, $n_1\times n_2$, $n_2\times n_1$, and $n_2\times n_2$ respectively, and Γ is $n_1\times n_2$. To see that (2) is a case of (1), take $m=2$ and

$$A = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}, \qquad Y(t) = \begin{pmatrix} \Gamma(t) \\ E_{n_2} \end{pmatrix}.$$

For other related results see [4].

2. Let X be $n\times n$ and such that

$$X' = AX. \tag{3}$$

The partitioning of A induces a partitioning of X into submatrices X_{ij} $(i, j=1, \cdots, k)$. Let $X_m=\mathrm{col}(X_{1m} \cdots X_{km})$. Then

$$X_m' = AX_m, \tag{4}$$

and, at least formally,

$$(X_{mm}^{-1})' = -X_{mm}^{-1}A_mX_{mm}^{-1}. \tag{5}$$

Thus, if X_{mm}^{-1} exists on some interval $I_0\subset I$, $X_mX_{mm}^{-1}$ is a solution on I_0 of (1). Further, if Y is a solution of the differential equation of (1), Y_m satisfies an equation of the form $P'+HP=H$; thus from classical

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existence and uniqueness theorems (1) has a local existence and uniqueness theorem and $Y_m(t) \equiv E_{n_m}$. Gathering together these remarks, we have

THEOREM 1. *Let $X(t)$ be the solution of* (3) *such that $X(t_0) = E_n$. Then the general solution near t_0 of* (1) *is $X(t)C(\sum_{i=1}^{k} X_{mi}C_i)^{-1}$, where C is constant and arbitrary, $C = \mathrm{col}(C_1 \cdots C_k)$, C_i is $n_i \times n_m$, and C_m is nonsingular.*

3. Now let $n = rk$ and $n_i = r$ $(i = 1, \cdots, k)$. We proceed to write the general solution of (1) in terms of solutions of (1) rather than solutions of (3). Let $U_1, \cdots, U_k$ denote solutions of (1); let $U = (U_1 \cdots U_k)$; let $Z = \mathrm{diag}(Z_j)$, where Z_j is $r \times r$ and

$$Z_j' = A_m U_j Z_j \qquad (j = 1, \cdots, k). \tag{6}$$

THEOREM 2. *Let $U(t_0)$ and $Z(t_0)$ be nonsingular; let $C = \mathrm{col}(C_j)$, the C_j being $r \times r$, constant, and arbitrary except that $\sum_{j=1}^{1} Z_j(t_0)C_j$ is nonsingular. Then the general solution of* (1) *can be written near t_0 as*

$$Y = UZC \cdot \left(\sum_{j=1}^{k} U_{mj} Z_j C_j \right)^{-1}. \tag{7}$$

PROOF. It is easily verified that UZ satisfies (3) and that, if $U(t_0)$ is nonsingular, $U(t)$ is nonsingular where it exists; the conclusion then follows.

Now let $U_{hij} = U_{hi} - U_{hj}$ and $V_{hij} = U_{hij}^{-1}$. It is easily verified that $(U_i - U_j)V_{hij}$ satisfies an equation of the form (1). Thus, if

$$[(U_i - U_q)V_{hiq} - (U_i - U_j)V_{hij}]_{t=t_0} = 0, \tag{8}$$

it is identically zero near t_0. Hence, if (8) is satisfied, $V_{hij}U_{hiq}$ satisfies

$$W' = A_m U_j W - W A_m U_q. \tag{9}$$

Now fix q; we may as well take $q = 1$.

THEOREM 3. *Let $|U(t_0)| \neq 0$. Let $i = k + j - 1$ $(j = 2, \cdots, k)$. Let U_i be a solution of* (1) *such that, for some r_j $(1 \leqq r_j \leqq k,\ r_j \neq m)$, $U_{r_j ij}$ and $U_{r_j i1}$ are nonsingular at t_0 and* (8) *is satisfied for $q = 1$ and $h = r_j$. Let $|Z_1(t_0)| \neq 0$. Then the general solution of* (1) *can be written in the form* (7), *where $Z_j = V_{r_j ij}U_{r_j i1}Z_1$ $(j = 2, \cdots, k)$.*

PROOF. Note that $Z_1' = A_m U_1 Z_1$ and that $V_{r_j ij}U_{r_j i1}$ satisfies (9) with $q = 1$; then apply Theorem 2.

We remark that, given q, j, and $h \neq k$, there does exist a set of initial values for the U_i which satisfies (8). However, if for U_j and

U_p $(p \neq j)$ the same initial values for U_i satisfy (8), there is a linear combination of columns of U_i, U_j, and U_p which is zero. Thus, since $|U(t_0)| \neq 0$, each U_j requires a distinct U_i. Thus, although we need not eliminate all of the Z_j's $(j=2, \cdots, k)$ in the general solution, we do need a distinct U_i for each one we do eliminate.

For $k=2$ we have Theorem 5 of [3]. For $r=1$ and general k we have a form for Y different from that given in [2].

Bibliography

1. A. Chiellini, *Sui sistemi di Riccati*, Rend. Sem. Fac. Sci. Univ. Cagliari vol. 18 (1948) pp. 44–58.

2. W. J. Coles, *Linear and Riccati systems*, Duke Math. J. vol. 22 (1955) pp. 333–338.

3. J. J. Levin, *On the matrix Riccati-equation*, Proc. Amer. Math. Soc. vol. 10 (1959) pp. 519–524.

4. W. T. Reid, *A matrix differential equation of Riccati-type*, Amer. J. Math. vol. 68 (1946) pp. 237–246.

University of Utah