ON COMPONENTS IN SOME FAMILIES OF SETS\textsuperscript{1}

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1. Introduction. Helly's well-known theorem [3] states that all the
members of a family of compact convex subsets of the Euclidean
n-space $E^n$ have a point in common provided every $n + 1$ members of
$\mathcal{C}$ have a common point. On the other hand (Motzkin, cf. Hadwiger-
Debrunner [2] for further reference), there exists no (finite) number
$h$ with the following property: If $\mathcal{K}$ is a family of subsets of $E^n$ (even
of $E^1$) such that each member of $\mathcal{K}$ is the union of at most two dis-
joint, compact, convex sets, and such that every $h$ members of $\mathcal{K}$
have a common point, then all the members of $\mathcal{K}$ have a common
point.

A consideration of the examples which establish the nonexistence
of $h$ led to the idea that there might exist theorems of Helly's type for
such families $\mathcal{K}$ if an additional condition is imposed on $\mathcal{K}$: the inter-
section of any two members of $\mathcal{K}$ should also be representable as
the union of at most two disjoint, compact, convex sets. The present
paper contains a theorem in this direction together with related re-
results on families $\mathcal{K}$ whose elements are disjoint unions of members of
another family $\mathcal{C}$.

In §2 we give the definitions of the properties we consider, and the
statements of our main results. The proofs follow in §3. Remarks, ex-
amples, and counter-examples are given in §4.

2. Definitions and results. We shall deal mainly with families of
subsets of some set, on whose nature nothing is assumed.

For a set $A$ or an ordinal $\mu$ we denote by $\text{card } A$ resp. $\text{card } \mu$ the
corresponding cardinal. Thus, for a family of sets $\mathcal{C} = \{C_\alpha : \alpha \in \mathcal{A}\}$
we have $\text{card } \mathcal{C} = \text{card } A$. The letter $\omega$ is used only for initial ordinals.

For a family of sets $\mathcal{C} = \{C_\alpha : \alpha \in \mathcal{A}\}$ we put $\pi C = \bigcap_{\alpha \in \mathcal{A}} C_\alpha$ and
$\sigma C = \bigcup_{\alpha \in \mathcal{A}} C_\alpha$.

We define $K = C_1 + C_2$ to be an abbreviation for the statement
"$K = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$." Similarly, for $\mathcal{C} = \{C_\alpha : \alpha \in \mathcal{A}\}$, we
write $K = \sum_{\alpha \in \mathcal{A}} C_\alpha = \Sigma C$ for "$K = \sigma C$ and $C_\alpha \cap C_\beta = \emptyset$ for all $\alpha, \beta \in \mathcal{A}$
with $\alpha \neq \beta$.

If $K = \Sigma \mathcal{C}$, each member of $\mathcal{C}$ is a component of $K$ and $\Sigma \mathcal{C}$ is a de-
composition of $K$.

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For any family $\mathcal{C}$ and any cardinal $\gamma$ let $[\mathcal{C}]_\gamma = \{ \Sigma \mathcal{C}' : \mathcal{C}' \subseteq \mathcal{C}, \ \text{card} \ \mathcal{C}' < \gamma + 1 \}$ and $[\mathcal{C}] = \{ \Sigma \mathcal{C}' : \mathcal{C}' \subseteq \mathcal{C} \}$. For $K \subseteq [\mathcal{C}]$ let $c(K) = \min \{ \text{card} \ \mathcal{C}' : K = \Sigma \mathcal{C}' , \ \mathcal{C}' \subseteq \mathcal{C} \}$.

This paper deals with some properties of families of sets which we proceed to define.

**Definition 1.** A family $\mathcal{C}$ is $\gamma$-intersectional (for a finite or infinite cardinal $\gamma \geq 1$) if for every subfamily $\mathcal{C}' \subseteq \mathcal{C}$ with $\text{card} \ \mathcal{C}' < \gamma + 1$ we have $\pi \mathcal{C}' \subseteq \mathcal{C}$. The family $\mathcal{C}$ is intersectional if it is $\gamma$-intersectional for every $\gamma \geq 1$.

Obviously, if $\gamma^* \leq \gamma$ and $\mathcal{C}$ is $\gamma$-intersectional, it is $\gamma^*$-intersectional as well. Every family is $1$-intersectional; every $2$-intersectional family is $\aleph_0$-intersectional.

**Definition 2.** A family $\mathcal{C}$ is $\gamma$-nonadditive (for a finite or infinite cardinal $\gamma \geq 2$) if for every subfamily $\mathcal{C}' \subseteq \mathcal{C}$, with $\emptyset \not\subseteq \mathcal{C}'$ and $1 < \text{card} \ \mathcal{C}' < \gamma + 1$, such that $\Sigma \mathcal{C}'$ is defined, we have $\Sigma \mathcal{C}' \not\subseteq \mathcal{C}$. The family $\mathcal{C}$ is nonadditive if it is $\gamma$-nonadditive for every $\gamma \geq 2$.

**Examples.** The family of all closed [open] subsets of $E^n$ is intersectional [$\aleph_0$-intersectional]. The family of all connected and open [compact] subsets of $E^n$ is nonadditive [$\aleph_1$-nonadditive; see [4]]. In the set of ordinals $\{ \alpha : \alpha < \omega , \text{card} \ \omega = k \}$, for any $k > \aleph_0$, all segments of the form $[\alpha , \beta]$ or $[\beta , \omega)$, where $\alpha , \beta$ are limit-ordinals, form a family $\mathcal{S}$ which is intersectional and nonadditive. For any set $S$ with card $S = k \geq \aleph_0$ the family of all subsets of $S$ with complements of cardinal less than $k$ is $\aleph_0$-intersectional and nonadditive.

**Definition 3.** A family $\mathcal{C}$ has the Helly property of order $h$ with limit $\gamma$ ($h$, $\gamma$ cardinals with $2 \leq h < \gamma$) if for each subfamily $\mathcal{C}' \subseteq \mathcal{C}$, with $\text{card} \ \mathcal{C}' < \gamma + 1$, the condition $"\pi \mathcal{C}' \neq \emptyset"$ for all $\mathcal{C}' \subseteq \mathcal{C}$, with $\text{card} \ \mathcal{C}' < h + 1$ implies $\pi \mathcal{C}' \neq \emptyset$. The family $\mathcal{C}$ has the unlimited Helly property of order $h$ if it has the Helly property of order $h$ with limit $\gamma$ for every $\gamma > h$.

**Examples.** The family of all compact subsets of any topological space has the unlimited Helly property of order $\aleph_0$. The family of convex subsets of $E^n$ has the Helly property of order $n + 1$ with limit $\aleph_0$; that of compact convex subsets has the unlimited Helly property of order $n + 1$ (Helly’s theorem). The family of all closed segments $[\alpha , \beta]$ of a well-ordered set has the unlimited Helly property of order $2$; if segments $[\alpha , \mu)$, for a limit ordinal $\mu$, are included, the family has the Helly property of order $2$ with limit card $\mu$.

The first theorem gives a criterion for the uniqueness of the decomposition of $K$.

**Theorem 1.** Let $\mathcal{C} = \{ C_\alpha : \alpha \in A \}$ be $2$-intersectional and $\gamma$-nonadditive, and $K \subseteq [\mathcal{C}]_\gamma$. If $K = \sum_{\alpha ' \in A '} C_\alpha'$ with $A ' \subseteq A$, card $A ' < \gamma + 1$,
and $C_{a'} \neq \emptyset$ for all $a' \in A'$, and if $K = \sum_{a'' \in A''} C_{a''}$ with $A'' \subset A$, card $A'' < \gamma + 1$, and $C_{a''} \neq \emptyset$ for all $a'' \in A''$, then there exists a one-to-one map $\phi$ from $A'$ onto $A''$ such that $C_{a'} = C_{\phi(a')}$ for all $a' \in A'$. In other words, the components of $K$ are uniquely determined.

As an immediate corollary we have:

**Corollary.** Let $\mathcal{C}$ be 2-intersectional and $\gamma$-nonadditive, and let $K \in [\mathcal{C}]_\gamma$ (i.e., $c(K) \leq n$), where $n$ is a finite cardinal and $\gamma \geq n$. Let $K^* \in [\mathcal{C}]_\gamma$, $K \subset K^*$, and let some $n$ different components of $K^*$ each have a nonempty intersection with $K$. Then different components of $K$ are contained in different components of $K^*$, and, in particular, $c(K) = n$.

Obvious examples show that the corollary may fail for infinite $n$.

The next theorem shows that $[\mathcal{C}]_\gamma$ is, in a sense, weakly intersectional: if the intersections of all members of certain subfamilies of $\mathcal{K} \subset [\mathcal{C}]$, belong to $[\mathcal{C}]_\gamma$, then for each subfamily of $\mathcal{K}$ the intersection of its members belongs to $[\mathcal{C}]_\gamma$.

**Theorem 2.** Let $\mathcal{C}$ be $\gamma$-intersectional and $\gamma'$-nonadditive, $\mathcal{K} \subset [\mathcal{C}]_\gamma$, and $\pi \mathcal{K} \subset [\mathcal{C}]_\gamma'$. Then there exists a subfamily $\mathcal{K}' \subset \mathcal{K}$, with $1 + \text{card } \mathcal{K}' \leq c(\pi \mathcal{K})$, such that different components of $\pi \mathcal{K}$ are contained in different components of $\pi \mathcal{K}'$; in particular, $c(\pi \mathcal{K}') \geq c(\pi \mathcal{K})$.

A result of Helly's type for members of $[\mathcal{C}]_2$ is given by

**Theorem 3.** Let $\mathcal{C}$ be $\gamma$-intersectional and $\mathcal{K}_0$-nonadditive, with the Helly property of order $h$ and limit $\gamma^*$, $\gamma^* \geq \mathcal{K}_0 > h$. Let $\mathcal{K} \subset [\mathcal{C}]_2$ be such that card $\mathcal{K} < \gamma + 1$ and $K' \cap K'' \subset [\mathcal{C}]_2$ for all $K', K'' \in \mathcal{K}$. Then $\mathcal{K}$ has the Helly property of order $2h$ with limit $\gamma^*$.

3. Proofs.

**Proof of Theorem 1.** Obviously

$$K = \sum_{a' \in A': a'' \in A''} (C_{a'} \cap C_{a''})$$

is a decomposition of $K$. If for each $a' \in A'$ and each $a'' \in A''$ either $C_{a'} \cap C_{a''} = \emptyset$ or $C_{a'} \cap C_{a''} = C_{a'}$, the theorem is proved. Assume on the contrary that there exists an $a' \in A'$ and an $a'' \in A''$ such that $C_{a'} \cap C_{a''}$ is neither $\emptyset$ nor $C_{a'}$. Let $A_{a'}' = \{a'' \in A'': C_{a'} \cap C_{a''} \neq \emptyset\}$. Then $2 \leq \text{card } A_{a'}' < \gamma + 1$ and $C_{a'} = C_{a'} \cap K = C_{a'} \cap \sum_{a'' \in A''} C_{a''} = \sum_{a'' \in A_{a'}'} (C_{a'} \cap C_{a''})$, in contradiction to the $\gamma$-nonadditivity of $\mathcal{C}$.

**Proof of Theorem 2.** (i) Let $c(\pi \mathcal{K}) \geq 2$. Then there exist points $x_1$ and $x_2$ contained in different components $C_1^*$, $C_2^*$ of $K^* = \pi \mathcal{K}$. For some $K_0 \in \mathcal{K}$ the points $x_1$ and $x_2$ are contained in different com-
ponents of $K_0$; indeed, otherwise there would for each $K \in \mathcal{K}$ exist a component $C'$ of $K$ with $x_1, x_2 \in C$. Now $C = \pi \{ C' : K \in \mathcal{K} \} \in \mathcal{E}$ but, on the other hand, $C = C \cap K^* \supseteq (C \cap C^*_1) + (C \cap C^*_2)$, and none of the components is empty (since $x_i \in C \cap C^*_i$), contradicting the $\gamma'$-nonadditivity of $\mathcal{E}$. If $c(K^*) = 2$, it follows at once from the corollary to Theorem 1 that different components of $K^*$ are contained in different components of $K_0$.

(ii) We now assume that $c(K^*) = n$ is finite, $n > 2$, and that the theorem is proved for all $n'$ with $n' < n$. We start as in (i) with a set $K_0 \equiv \sum_{v \in N} C_v \in \mathcal{K}$, where $\text{card} \, N = c(K_0) \geq 2$, such that $C_1 \cap K^* \neq \emptyset \text{ and } C_2 \cap K^* \neq \emptyset$. Let $q_v = c(K^* \cap C_v) \geq 0$ for $v \in N$. By Theorem 1 we have

\[
\sum_{v \in N} q_v = c(K^*) = n.
\]

This implies that $N_0 = \{ v \in N : q_v > 0 \}$ is finite and contains at most $n$ elements. Let us assume that $N_0 = \{ 1, 2, \ldots, t \}$ and that the components of $K_0$ are labeled in such a way that $q_1 \geq 2$ for $1 \leq v \leq s$, and $q_v = 1$ for $s < v \leq t$. If $s = 0$, then (*) implies $t = n$, and by the corollary to Theorem 1 the $n$ components of $K_0$ contain the $n$ components of $K^*$, as claimed. Thus we are left with the case $s \geq 1$; then $2 \leq t < n$, $q_1 \geq 2$ and, by the choice of $K_0$, $q_2 \geq 1$; therefore, by (*), $q_v < n$ for all $v \in N_0$. This allows us to apply the inductive assumption to each of the $s$ families $\mathcal{K}_v = \{ C_v \cap K : K \in \mathcal{K} \}$, $1 \leq v \leq s$. It follows that for each $v$, with $1 \leq v \leq s$, there exists a subfamily $\mathcal{K}' \subset \mathcal{K}_v$, containing $p_v \leq q_v - 1$ members, such that the different components of $C_v \cap K^*$ are contained in different components of $\pi \mathcal{K}'$. The family $\mathcal{K}' = \{ K_0 \}

\cup (U_{v=1}^t \mathcal{K}_v')$ satisfies all the conditions of the theorem. Indeed, different components of $K^*$ are, by the corollary to Theorem 1, contained in different components of $\pi \mathcal{K}'$; but on the other hand, $\mathcal{K}'$ contains only $1 + \sum_{v=1}^t p_v \leq 1 - s + \sum_{v=1}^s q_v = 1 - s + n - (t - s) = n + 1 - t \leq n - 1 < c(K^*)$ members.

(iii) There remains the case in which $k = c(K^*)$ is infinite. Let $\omega$ be the initial ordinal of $k$ and let $K^* = \pi \mathcal{K} = \sum_{v < \omega} C^*_v$. For each $v < \omega$ let $x_v \in C^*_v$. As in (i), for each pair $v, \mu < \omega$ with $v \neq \mu$ there exists some $K_{v,}\mu \in \mathcal{K}$ such that $x_v$ and $x_\mu$ are contained in different components of $K_{v,}\mu$. Let $\mathcal{K}' = \{ K_{v,}\mu : x_v, \mu < \omega \}$. Then $\text{card} \, \mathcal{K}' \equiv (\text{card} \, \omega)^2 = k$. For the family $\mathcal{K}'$ we have $c(\pi \mathcal{K}') \geq k$ since $x_v$ and $x_\mu$ belong to different components of $\pi \mathcal{K}'$. By an argument similar to that used in the proof of Theorem 1 it follows that different components of $K^*$ are contained in different components of $\pi \mathcal{K}'$. This ends the proof of Theorem 2.
Proof of Theorem 3. For some fixed \( h \) assume the theorem false; let \( k = h \) be the minimal cardinal for which there exists a family with \( \mathcal{K} = k \) contradicting the theorem.

(i) Assume \( k \) finite. Then for each \( K^* \in \mathcal{K} \) we have \( \pi \{ K \in \mathcal{K} : K \neq K^* \} \neq \emptyset \). Let \( \mathcal{X}_i = \{ K \in \mathcal{K} : c(K) = i \} \) for \( i = 1, 2 \), and let \( K = C_1 + C_2 \) for all \( K \in \mathcal{K}_2 \). We assume that \( \mathcal{K} \) is chosen in such a way that card \( \mathcal{X}_1 + 2 \) card \( \mathcal{X}_2 \) (the total number of components of members of \( \mathcal{K} \)) is minimal. This implies that for each \( K' \in \mathcal{K}_2 \) and \( i = 1, 2 \), there exists a \( K^0 = K^0(C_i') = K^0(K', i) \in \mathcal{K} \) such that \( \pi \{ K \in \mathcal{K} : K \neq K^0 \} \subset C_i' \).

We shall show that \( C_i' \cap K \neq \emptyset \) for all \( K' \in \mathcal{K}_2 \), \( K \in \mathcal{K} \), and \( i = 1, 2 \). Let us assume, to the contrary, that there exists \( K' \in \mathcal{K}_2 \), \( K \in \mathcal{K} \), and \( i = 1 \) or 2 such that \( C_i' \cap K = \emptyset \). (Without loss of generality we shall assume \( i = 1 \).) Since \( \emptyset \neq \pi \{ K \in \mathcal{K} : K \neq K^0(K', 1) \} \subset C_1' \), it follows that \( K_0 = K^0(K', 1) \). Then \( C_1' \cap K = \emptyset \) for all \( K \neq K^0 \); also \( C_i' \cap K \neq \emptyset \) for all \( K \in \mathcal{K} \), since otherwise \( K' \cap K \cap K_0 \subset (C_i' \cap K_0) \cup (C_i' \cap K) = \emptyset \) would contradict the assumption that any \( 3 < 4 \leq 2h \) members of \( \mathcal{K} \) have a nonempty intersection. Therefore, for each \( K \neq K_0 \), \( c(K' \cap K) = 2 \); hence, for some component \( C_j \) of \( K \) we have \( K \cap C_j = C_j \cap C_j' \). Now

\[
\pi \{ C_j : K \in \mathcal{K}, K \neq K_0 \} = C_1' \cap \pi \{ C_j : K \in \mathcal{K}, K \neq K_0 \} = C_1' \cap \pi \{ K \in \mathcal{K} : K \neq K_0 \} \subset C_1' \cap C_1' = \emptyset.
\]

Since \( \mathcal{C} \) has the Helly property of order \( h \) it follows that for some subset \( \mathcal{K}_0 \) of \( \mathcal{K} \), such that \( K_0 \in \mathcal{K}_0 \) and with card \( \mathcal{K}_0 = h_0 \leq h \), we have \( \pi \{ C_j : K \in \mathcal{K}_0 \} = \emptyset \). For the family \( \mathcal{K}^* = \{ K', K_0 \} \cup \mathcal{K}_0 \) we have therefore \( \pi \mathcal{K}^* \subset (C_1' \cap K') \cup (C_1' \cap \pi \mathcal{K}_0) = \emptyset \), although card \( \mathcal{K}^* \leq h_0 + 2 \leq h + 2 \leq 2h \). This contradiction establishes our assertion.

Next, let \( K^* \in \mathcal{K}_2 \) be chosen arbitrarily. For each \( K \in \mathcal{K}_2 \) it follows from the above and from \( c(K' \cap K) \leq 2 \) that \( c(K^* \cap K) = 2 \) and that different components of \( K \) intersect different components of \( K^* \). Let the components of \( K \) be re-labeled, if necessary, in such a way that \( C_i' \cap C_i' \neq \emptyset \) for \( i = 1, 2 \). We claim that for all \( K', K'' \in \mathcal{K}_2 \) we have \( C_1' \cap C_1'' \neq \emptyset, i = 1, 2 \). Indeed, otherwise we would have (since each component of one set intersects every other set), \( C_1' \cap C_1'' = C_1' \cap C_1'' = \emptyset \), and therefore \( K^* \cap K' \cap K'' = \emptyset \), which is impossible. Thus, for any \( K', K'' \in \mathcal{K}_2 \),

\[
C_1' \cap C_1'' = \begin{cases} \emptyset & \text{if } i \neq j, \\ \neq \emptyset & \text{if } i = j. \end{cases}
\]

Now we consider the families \( \mathcal{C}_i = \mathcal{K}_1 \cup \{ C_j : K \in \mathcal{K}_2 \} \) for \( i = 1, 2 \). The assumption \( \pi \mathcal{K} = \emptyset \) implies that \( \pi \mathcal{C}_i = \emptyset \) for \( i = 1, 2 \). Since \( \mathcal{C}_1 \subset \mathcal{C}_2 \),
there exist \( h \) or less members of \( \mathcal{C}_i \) whose intersection is empty, \( i = 1, 2 \). But then the intersection of the corresponding members of \( \mathcal{K} \) is also empty, although it involves at most \( 2h \) members of \( \mathcal{K} \). The contradiction reached proves the theorem for finite \( k \).

(ii) Let \( k \) be infinite, \( k < \gamma^* \), and the theorem true for all families with less than \( k \) members. Let \( \omega \) be the initial ordinal of \( k \), let \( A \) be the set of ordinals \( A = \{ \alpha : \alpha < \omega \} \), and let \( \mathcal{K} = \{ K_\alpha : \alpha < \omega \} \). By the inductive assumption we have \( \cap_{\alpha < \mu} K_\alpha \neq \emptyset \) for each \( \mu < \omega \). If for some \( K_\alpha \) one of its components does not intersect some \( K_\beta \), we omit this component and take the other component to be the new \( K_\alpha \). By the inductive assumption, the new \( K_\alpha \) satisfy \( \cap_{\alpha < \mu} K_\alpha \neq \emptyset \) for all \( \mu < \omega \). From here on we proceed as in the final part of (i): we re-label (if necessary) the components of some \( K_\alpha \) with \( c(K_\alpha) = 2 \), construct the families \( \mathcal{C}_i \) and derive a contradiction from the assumption that \( \cap_{\alpha < \omega} K_\alpha = \emptyset \). This terminates the proof of Theorem 3.

4. Remarks. 1. Theorem 2 fails if \( \text{card} \, \pi \mathcal{K} \) is infinite and \( \mathcal{K}' \) is assumed to satisfy \( \text{card} \, \mathcal{K}' < \text{card} \, \pi \mathcal{K} \). E.g., starting from the family \( \mathcal{S} \) (preceding Definition 3), with \( \text{card} \, \omega = k > \aleph_0 = \text{card} \, \omega_0 \), let \( \mathcal{K} = \{ [\omega_0, \alpha] \cup [\alpha + \omega_0, \omega) : \alpha \text{ limit ordinal } < \omega \} \). Then \( c(\pi \mathcal{K}) = k \), but the intersection of any \( k' < k \) members of \( \mathcal{K} \) has only \( k' \) components. Similar examples are easily found for \( c(\pi \mathcal{K}) = \aleph_0 \).

2. Probably the most interesting immediate application of Theorem 3 is to convex sets in \( E^2 \). To satisfy the condition of nonadditivity we may consider, e.g., families consisting only of closed (or only of open) convex sets. The following example shows that Theorem 3 does not hold if \( \mathcal{C} \) is, e.g., the family of all convex sets in \( E^2 \). (Simple examples of a similar nature show the necessity of nonadditivity assumptions in Theorem 2.) Let \( D \) denote a closed disc with center 0. Let \( K_0 \) be obtained from \( D \) by deleting 0. Let \( x_i, i = 1, 2, \ldots, 6 \), be equidistant points on the boundary of \( D \), \( x_i = x_{i+6} \). For each \( i, 1 \leq i \leq 6 \), let \( K_i \) be obtained from \( D \) by deleting the open small arc of \( \text{Bd} \, D \) determined by \( x_{i-1} \) and \( x_{i+1} \), and the open sector determined by these two points and 0. Then each \( K_i, 0 \leq i \leq 6 \), as well as the intersection of any two \( K_i \), is the disjoint union of two convex sets, and any six \( K_i \) have a nonempty intersection. Nevertheless, \( \cap_{i=0}^6 K_i = \emptyset \). As is easily verified, the same reasoning applies to the case where 7 or 8 equidistant points are chosen on \( \text{Bd} \, D \). We conjecture that for the family of all convex sets in \( E^2 \) a result analogous to Theorem 3 holds, with 9 instead of \( 2h \).

3. The following statement (with obvious refinements) is conjectured: If \( \mathcal{C} \) is an intersectional and nonadditive family with un-
limited Helly property of order \( h \) and if \( \mathcal{K} \subseteq [\mathcal{C}]_n \) is such that the intersection of any 2, 3, \( \ldots \), \( n \) members of \( \mathcal{K} \) also belongs to \( [\mathcal{C}]_n \), then \( \mathcal{K} \) has the unlimited Helly property of order \( nh \). Simple examples show that \( nh - 1 \) may not be substituted for \( nh \) in this conjecture. If \( \mathcal{C} \) is the family of segments in \( E^1 \), the conjecture is easily provable.

4. Let \( \mathcal{C}^{(n)} \) denote the family of all compact, convex subsets of \( E^n \); in \([1]\), a function \( \Delta(K) \), with \( 0 \leq \Delta(K) \leq +\infty \), was defined for all compact sets \( K \subseteq E^n \) in such a way that \( \Delta(K) < \infty \) if and only if \( K \subseteq [\mathcal{C}^{(n)}]_{K'} \). Theorem 2 of \([1]\) may be formulated as follows: For any finite \( n \geq 1 \) and real \( d < \infty \) there exists a finite \( h = h(n, d) \) such that the family \( \{ K \subseteq [\mathcal{C}^{(n)}]_{K'} : \Delta(K) \leq d \} \) has the unlimited Helly property of order \( h \). By applying the methods of \([1]\) it may be shown that for each finite \( n \geq 1 \) and \( d < \infty \) there exists a finite \( k = k(n, d) \) such that \( \Delta(K) \leq d \) implies \( K \subseteq [\mathcal{C}^{(n)}]_k \).

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