# ON THE ORDER OF INTEGRAL FUNCTIONS DEFINED BY DIRICHLET SERIES

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1. Consider the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$$

where $\lambda_{n+1} > \lambda_n$, $\lambda_1 \geqq 0$, $\lim_{n\to\infty} \lambda_n = \infty$, $s = \sigma + it$ and

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0. \tag{1.1}$$

Let $\sigma_e$ and $\sigma_a$ be the abscissa of convergence and the abscissa of absolute convergence, respectively, of $f(s)$. If $\sigma_e = \sigma_a = \infty$, $f(s)$ is an integral function. We shall suppose throughout *that* (1.1) *holds and that* $\sigma_e = \sigma_a = \infty$.

It is known that [1, p. 67]

$$\log \mu(\sigma) = \log \mu(\sigma_0) + \int_{\sigma_0}^{\sigma} \lambda_{\nu(t)} dt, \tag{1.2}$$

where $\nu(\sigma)$ gives the rank of the maximum term

$$\mu(\sigma) = \max_{n\geqq 1} \{ | a_n | e^{\sigma\lambda_n} \}$$

and

$$\mu(\sigma) \leqq M(\sigma),$$

where $M(\sigma) = \max_{-\infty<t<\infty} |f(\sigma+it)|$. Also, if $\rho$ be the Ritt-order [2 p. 78] of $f(s)$ then [1, p. 69]

$$\limsup_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log | a_n |^{-1}} = \rho = \limsup_{\sigma\to\infty} \frac{\log \log M(\sigma)}{\sigma}. \tag{1.3}$$

A corresponding result for the lower order $\lambda$ does not always hold. In fact, it has been shown that, if $\log \lambda_n \sim \log \lambda_{n+1}$ then [3, p. 97]

$$\lambda \geqq \liminf_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log | a_n |^{-1}}, \tag{1.4}$$

where

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$$\liminf_{\sigma\to\infty} \frac{\log\log M(\sigma)}{\sigma} = \lambda \qquad (0 \leqq \lambda \leqq \infty).$$

But, if $\log |a_n/a_{n+1}|/(\lambda_{n+1}-\lambda_n)$ is a nondecreasing function of $n$ for $n>n_0$, then [3, p. 97]

$$\lambda \leqq \liminf_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}}. \tag{1.5}$$

(1.4) and (1.5) lead to the following:

THEOREM A. *If $f(s)=\sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ is an integral function such that* (i) $\log \lambda_n \sim \log \lambda_{n+1}$, (ii) $\log |a_n/a_{n+1}|/(\lambda_{n+1}-\lambda_n)$ *form a nondecreasing function of $n$ for $n>n_0$, then*

$$\liminf_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}} = \lambda \qquad (0 \leqq \lambda \leqq \infty). \tag{1.6}$$

The integral function $f(s)$ is said to be of linearly regular growth if $\rho=\lambda$. In this paper we derive a few properties of integral functions defined by Dirichlet series in relation to order and the ranks of their maximum terms.

2. THEOREM 1. *If $f_1(s)=\sum_{n=1}^{\infty} a_{1,n}e^{s\lambda_{1,n}}$ and $f_2(s)=\sum_{n=1}^{\infty} a_{2,n}e^{s\lambda_{2,n}}$ be integral functions of orders $\rho_1$ $(0<\rho_1<\infty)$, $\rho_2$ $(0<\rho_2<\infty)$, then the function $f(s)=\sum_{n=1}^{\infty} a_n e^{s\lambda_n}$, where* (i) $\lambda_{1,n}\sim\lambda_{2,n}\sim\lambda_n$ *and* (ii) $|a_n| \sim |a_{1,n}|\,|a_{2,n}|$, *is an integral function of order $\rho$ such that*

$$1/\rho \geqq 1/\rho_1 + 1/\rho_2. \tag{2.1}$$

*Further, if $\gamma_1$ $(0<\gamma_1<\infty)$, $\gamma_2$ $(0<\gamma_2<\infty)$ be the lower orders of $f_1(s)$ and $f_2(s)$ respectively, and* (iii) $\log \lambda_n \sim \log \lambda_{n+1}$, (iv) $\log |a_{1,n}/a_{1,n+1}|/(\lambda_{1,n+1}-\lambda_{1,n})$ *and* $\log |a_{2,n}/a_{2,n+1}|/(\lambda_{2,n+1}-\lambda_{2,n})$ *nondecreasing functions of $n$ for $n>n_0$, then*

$$1/\lambda \leqq 1/\gamma_1 + 1/\gamma_2, \tag{2.2}$$

*where $\lambda$ is the lower order of $f(s)$.*

COROLLARY. *If the conditions* (i)–(iv) *hold and both $f_1(s)$ and $f_2(s)$ be of linearly regular growth then so is the function $f(s)$ and*

$$1/\rho = 1/\rho_1 + 1/\rho_2.$$

PROOF. Since $\lambda_{1,n}\sim\lambda_{2,n}\sim\lambda_n$, it is evident that

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = \limsup_{n\to\infty} \frac{\log n}{\lambda_{1,n}} = \limsup_{n\to\infty} \frac{\log n}{\lambda_{2,n}} = 0,$$

since both $f_1(s)$ and $f_2(s)$ satisfy (1.1) by hypothesis. Further, since $f_1(s)$ and $f_2(s)$ are integral functions, bounded in $\sigma < x$ for any $x < \infty$, the series $\sum_{n=1}^{\infty} |a_{1,n}| e^{\sigma\lambda_{1,n}}$ and $\sum_{n=1}^{\infty} |a_{2,n}| e^{\sigma\lambda_{2,n}}$ are convergent for every $\sigma$, and as $\lambda_n \sim \lambda_{1,n} \sim \lambda_{2,n}$ we shall have

$$\sum_{n=1}^{\infty} | a_{1,n} | e^{\sigma\lambda_n} < \infty$$

and

$$\sum_{n=1}^{\infty} | a_{2,n} | e^{\sigma\lambda_n} < \infty$$

for every $\sigma$. Hence, it follows that $\lim_{n\to\infty} |a_{1,n}| = 0$ and $\sum_{n=1}^{\infty} |a_{1,n}| \, |a_{2,n}| e^{\sigma\lambda_n} < \infty$ and, since $|a_n| \sim |a_{1,n}| \, |a_{2,n}|$, we have $\sum_{n=1}^{\infty} |a_n| e^{\sigma\lambda_n} < \infty$ for every $\sigma$. Hence, $f(s)$ is an integral function.

Again, using (1.3) for $f_1(s)$ and $f_2(s)$, we get

$$\liminf_{n\to\infty} \frac{-\log | a_{1,n} |}{\lambda_n \log \lambda_n} = 1/\rho_1,$$

since $\lambda_n \sim \lambda_{1,n}$ and

$$\liminf_{n\to\infty} \frac{-\log | a_{2,n} |}{\lambda_n \log \lambda_n} = 1/\rho_2,$$

since $\lambda_n \sim \lambda_{2,n}$. Therefore, for any $\epsilon > 0$, we have for sufficiently large $n$,

$$\frac{-\log | a_{1,n} |}{\lambda_n \log \lambda_n} > 1/\rho_1 - \epsilon/2,$$

and

$$\frac{-\log | a_{2,n} |}{\lambda_n \log \lambda_n} > 1/\rho_2 - \epsilon/2.$$

Or,

$$\frac{-\log | a_{1,n} | \, | a_{2,n} |}{\lambda_n \log \lambda_n} > 1/\rho_1 + 1/\rho_2 - \epsilon.$$

And, since $|a_n| \sim |a_{1,n}| \, |a_{2,n}|$, we get,

$$\liminf_{n\to\infty} \frac{-\log | a_n |}{\lambda_n \log \lambda_n} = \liminf_{n\to\infty} \frac{-\log | a_{1,n} | \, | a_{2,n} |}{\lambda_n \log \lambda_n} \geqq 1/\rho_1 + 1/\rho_2.$$

Or,

$$1/\rho \geqq 1/\rho_1 + 1/\rho_2.$$

Again, if $\log \lambda_n \sim \log \lambda_{n+1}$, we have $\log \lambda_{1,n} \sim \log \lambda_{1,n+1}$ and $\log \lambda_{2,n} \sim \log \lambda_{2,n+1}$ in view of (i). Hence, if (iv) also holds, (1.6) of Theorem A gives

$$\limsup_{n\to\infty} \frac{-\log|a_{1,n}|}{\lambda_n \log \lambda_n} = 1/\gamma_1,$$

for the function $f_1(s)$. Therefore, for any $\epsilon > 0$ and $n > n_1 \geqq n_0$, we have

$$\frac{-\log|a_{1,n}|}{\lambda_n \log \lambda_n} < 1/\gamma_1 + \epsilon/2. \tag{2.3}$$

Similarly, for $f_2(s)$ we have for $n > n_2 \geqq n_0$

$$\frac{-\log|a_{2,n}|}{\lambda_n \log \lambda_n} < 1/\gamma_2 + \epsilon/2. \tag{2.4}$$

Combining (2.3) and (2.4), we get

$$\frac{-\log|a_{1,n}|\,|a_{2,n}|}{\lambda_n \log \lambda_n} < 1/\gamma_1 + 1/\gamma_2 + \epsilon$$

for sufficiently large n. As $|a_n| \sim |a_{1,n}|\,|a_{2,n}|$, this gives

$$\limsup_{n\to\infty} \frac{-\log|a_n|}{\lambda_n \log \lambda_n} = \limsup_{n\to\infty} \frac{-\log|a_{1,n}|\,|a_{2,n}|}{\lambda_n \log \lambda_n} \leqq 1/\gamma_1 + 1/\gamma_2.$$

Or,

$$1/\lambda \leqq 1/\gamma_1 + 1/\gamma_2.$$

The corollary follows from (2.1) and (2.2) since for functions of linearly regular growth, $\rho_1 = \gamma_1$, $\rho_2 = \gamma_2$ and so

$$1/\lambda \leqq 1/\gamma_1 + 1/\gamma_2 = 1/\rho_1 + 1/\rho_2 \leqq 1/\rho.$$

But $\rho \geqq \lambda$ and so $1/\lambda = 1/\rho_1 + 1/\rho_2 = 1/\rho$, which gives $\rho = \lambda$. Hence, $f(s)$ is also of linearly regular growth.

THEOREM 2. *If* $f_1(s) = \sum_{n=1}^{\infty} a_{1,n} e^{s\lambda_{1,n}}$ *and* $f_2(s) = \sum_{n=1}^{\infty} a_{2,n} e^{s\lambda_{2,n}}$ *be integral functions of linearly regular growth and such that* (i) $\lambda_{1,n} \sim \lambda_{2,n} \sim \lambda_n$, (ii) $\log \lambda_n \sim \log \lambda_{n+1}$ *and* (iii) $\log |a_{1,n}/a_{1,n+1}|/(\lambda_{1,n+1} - \lambda_{1,n})$, $\log |a_{2,n}/a_{2,n+1}|/(\lambda_{2,n+1} - \lambda_{2,n})$ *nondecreasing functions of* $n$ *for* $n > n_0$, *then* $f_1(s), f_2(s)$ *will be of the same finite order* $\rho$, *if and only if*

$$\log\{|a_{1,n}|/|a_{2,n}|\} = o(\lambda_n \log \lambda_n) \tag{2.5}$$

*as* $n \to \infty$.

PROOF. Since $f_1(s)$ satisfies the conditions of Theorem A and is also of regular growth, from (1.3) and (1.6), we have

$$\limsup_{n\to\infty} \frac{\lambda_{1,n} \log \lambda_{1,n}}{\log |a_{1,n}|^{-1}} = \rho = \lambda = \liminf_{n\to\infty} \frac{\lambda_{1,n} \log \lambda_{1,n}}{\log |a_{1,n}|^{-1}}.$$

Further, since $\lambda_{1,n} \sim \lambda_n$, this gives

$$\lim_{n\to\infty} \frac{-\log |a_{1,n}|}{\lambda_n \log \lambda_n} = 1/\rho. \tag{2.6}$$

Similarly, for the function $f_2(s)$, if it is of order $\rho$, we have

$$\lim_{n\to\infty} \frac{-\log |a_{2,n}|}{\lambda_n \log \lambda_n} = 1/\rho. \tag{2.7}$$

Thus, (2.5) follows on subtracting (2.6) from (2.7).

Again, if $\rho_1$, $\rho_2$ be the orders of $f_1(s)$, $f_2(s)$, we have

$$1/\rho_2 - 1/\rho_1 = \lim_{n\to\infty} \frac{\log\{ |a_{1,n}| / |a_{2,n}| \}}{\lambda_n \log \lambda_n} = 0,$$

if (2.5) holds. Hence, $\rho_2 = \rho_1$.

3. We shall now derive a result involving the ranks of the maximum terms in the integral function $f(s)$ and its $n$th derivative $f^{(n)}(s)$. First we shall prove a lemma.

LEMMA. *If $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ be an integral function of linear order $\rho$ and lower linear order $\lambda$, $\nu(\sigma, f)$, $\nu(\sigma, f^{(n)})$ denote the ranks of the maximum terms $\mu(\sigma, f)$ and $\mu(\sigma, f^{(n)})$, for* $\mathrm{Re}(s) = \sigma$, *in the series for $f(s)$ and its nth derivative $f^{(n)}(s)$, respectively, then*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix}\left[\frac{1}{\sigma}\int_{\sigma_0}^{\sigma} \chi(t, n)dt\right] = \begin{matrix} n\rho, \\ n\lambda \end{matrix} \tag{3.1}$$

*where* $\chi(\sigma, n) = \lambda_{\nu(\sigma, f^{(n)})} - \lambda_{\nu(\sigma, f)}$, $n = 1, 2, \cdots$.

PROOF. Using (1.2) for $f(s)$ and $f^{(n)}(s)$, we have

$$\log \mu(\sigma, f) = \int_{\sigma_0}^{\sigma} \lambda_{\nu(t,f)} dt + O(1)$$

and

$$\log \mu(\sigma, f^{(n)}) = \int_{\sigma_0}^{\sigma} \lambda_{\nu(t,f^{(n)})} dt + O(1).$$

Therefore,

$$\log \frac{\mu(\sigma, f^{(n)})}{\mu(\sigma, f)} = \int_{\sigma_0}^{\sigma} [\lambda_{\nu(t,f^{(n)})} - \lambda_{\nu(t,f)}]dt + O(1).$$

Hence,

$$\frac{\log \dfrac{\mu(\sigma, f^{(n)})}{\mu(\sigma, f)}}{\sigma} = \frac{1}{\sigma} \cdot \int_{\sigma_0}^{\sigma} \chi(t, n)dt + o(1),$$

and (3.1) follows, since [4, p. 89]

$$\lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \left[ \frac{\log \dfrac{\mu(\sigma, f^{(n)})}{\mu(\sigma, f)}}{\sigma} \right] = \begin{matrix} n\rho, \\ n\lambda. \end{matrix}$$

The following results are the direct consequences of the above lemma:

(i) *For any* $\epsilon > 0$, *we can find a* $\sigma_0$ *such that*

$$(n\lambda - \epsilon)\sigma < \int_{\sigma_0}^{\sigma} \chi(t, n)dt < (n\rho + \epsilon)\sigma, \tag{3.2}$$

*for* $\sigma > \sigma_0$.

(ii) *If* $f(s)$ *is of linearly regular growth* $(\rho = \lambda)$,

$$\int_{\sigma_0}^{\sigma} \chi(t, n)dt \sim n\rho\sigma \tag{3.3}$$

*as* $\sigma \to \infty$.

THEOREM 3. *If* $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ *is an integral function of linearly regular growth and* $\lim_{\sigma \to \infty} \chi(\sigma, n)$ *exists then* $f(s)$ *is of finite order* $\rho$ *such that*

$$\lim_{\sigma \to \infty} \chi(\sigma, n) = n\rho, \tag{3.4}$$

*where* $\chi(\sigma, n) = \lambda_{\nu(\sigma,f^{(n)})} - \lambda_{\nu(\sigma,f)}$, *for* $n = 1, 2, \cdots, \nu(\sigma, f)$ *and* $\nu(\sigma, f^{(n)})$ *being the ranks of the maximum terms in* $f(s)$ *and its nth derivative* $f^{(n)}(s)$ *respectively.*

PROOF. We have

$$\begin{aligned} \mu(\sigma, f) &= \left| a_{\nu(\sigma,f)} \right| \exp(\sigma\lambda_{\nu(\sigma,f)}) \\ &\geqq \left| a_{\nu(\sigma,f^{(1)})} \right| \exp(\sigma\lambda_{\nu(\sigma,f^{(1)})}) \\ &= \mu(\sigma, f^{(1)})/\lambda_{\nu(\sigma,f^{(1)})}, \end{aligned}$$

since $\mu(\sigma, f^{(1)}) = \lambda_{\nu(\sigma,f^{(1)})} \left| a_{\nu(\sigma,f^{(1)})} \right| \exp(\sigma\lambda_{\nu(\sigma,f^{(1)})})$. Also, $\mu(\sigma, f^{(1)}) \geqq \lambda_{\nu(\sigma,f)} \left| a_{\nu(\sigma,f)} \right| \exp(\sigma\lambda_{\nu(\sigma,f)}) = \lambda_{\nu(\sigma,f)}\mu(\sigma, f)$. Hence,

$$\lambda_{\nu(\sigma,f)} \leqq \frac{\mu(\sigma, f^{(1)})}{\mu(\sigma, f)} \leqq \lambda_{\nu(\sigma,f^{(1)})}.$$

Using the above inequalities for the functions $f^{(1)}(s), f^{(2)}(s), \cdots$, etc., we get

$$\lambda_{\nu(\sigma,f)} \leqq \lambda_{\nu(\sigma,f^{(1)})} \leqq \lambda_{\nu(\sigma,f^{(2)})} \leqq \cdots$$

so that $\chi(\sigma, n) = \lambda_{\nu(\sigma,f^{(n)})} - \lambda_{\nu(\sigma,f)} \geqq 0$, for $n = 1, 2, \cdots$. Now, if $\lim_{\sigma\to\infty} \chi(\sigma, n) = l$, say, we have for any $\epsilon > 0$ and $\sigma > \sigma_0$,

$$l - \epsilon < \chi(\sigma, n) < l + \epsilon.$$

Hence,

$$\frac{(l-\epsilon)}{\sigma}\int_{\sigma_0}^{\sigma} dt < \frac{1}{\sigma}\int_{\sigma_0}^{\sigma} \chi(t, n)dt < \frac{(l+\epsilon)}{\sigma}\int_{\sigma_0}^{\sigma} dt,$$

or,

$$(l-\epsilon)(1 - o(1)) < \frac{1}{\sigma}\int_{\sigma_0}^{\sigma} \chi(t, n)dt < (l+\epsilon)(1 - o(1)).$$

Therefore, on proceeding to limits, we get

$$\lim_{\sigma\to\infty}\left[\frac{1}{\sigma}\int_{\sigma_0}^{\sigma} \chi(t, n)dt\right] = l = n\rho,$$

in view of (3.3).

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