ON THE ORDER OF INTEGRAL FUNCTIONS DEFINED 
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1. Consider the Dirichlet series 

\[ f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n} \]

where \( \lambda_{n+1} > \lambda_n, \lambda_1 \geq 0, \lim_{n \to \infty} \lambda_n = \infty, s = \sigma + it \) and 

\[ \limsup_{n \to \infty} \frac{\log n}{\lambda_n} = 0. \]  

(1.1)

Let \( \sigma_e \) and \( \sigma_a \) be the abscissa of convergence and the abscissa of absolute convergence, respectively, of \( f(s) \). If \( \sigma_e = \sigma_a = \infty \), \( f(s) \) is an integral function. We shall suppose throughout that \( (1.1) \) holds and that \( \sigma_e = \sigma_a = \infty \).

It is known that [1, p. 67] 

\[ \log \mu(\sigma) = \log \mu(\sigma_0) + \int_{\sigma_0}^{\sigma} \nu(t) dt, \]

where \( \nu(\sigma) \) gives the rank of the maximum term 

\[ \mu(\sigma) = \max_{n \geq 1} \{ |a_n| e^{\lambda_n} \} \]

and 

\[ \mu(\sigma) \leq M(\sigma), \]

where \( M(\sigma) = \max_{-\infty < t < \infty} |f(\sigma + it)| \). Also, if \( \rho \) be the Ritt-order [2 p. 78] of \( f(s) \) then [1, p. 69] 

\[ \limsup_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}} = \rho = \limsup_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\sigma}. \]  

(1.3)

A corresponding result for the lower order \( \lambda \) does not always hold. In fact, it has been shown that, if \( \log \lambda_n \sim \log \lambda_{n+1} \) then [3, p. 97] 

\[ \lambda \geq \liminf_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}}, \]

where

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\[
\lim_{\sigma \to -\infty} \inf \frac{\log \log M(\sigma)}{\sigma} = \lambda \quad (0 \leq \lambda \leq \infty).
\]

But, if \( \log |a_n/a_{n+1}|/(\lambda_{n+1} - \lambda_n) \) is a nondecreasing function of \( n \) for \( n>n_0 \), then [3, p. 97]

\[
(1.5) \quad \lambda \leq \liminf_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}}.
\]

(1.4) and (1.5) lead to the following:

**THEOREM A.** If \( f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s} \) is an integral function such that

(i) \( \log \lambda_n \sim \log \lambda_{n+1} \), (ii) \( \log |a_n/a_{n+1}|/(\lambda_{n+1} - \lambda_n) \) form a nondecreasing function of \( n \) for \( n>n_0 \), then

\[
(1.6) \quad \liminf_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}} = \lambda \quad (0 \leq \lambda \leq \infty).
\]

The integral function \( f(s) \) is said to be of linearly regular growth if \( \rho = \lambda \). In this paper we derive a few properties of integral functions defined by Dirichlet series in relation to order and the ranks of their maximum terms.

2. **Theorem 1.** If \( f_1(s) = \sum_{n=1}^{\infty} a_{1,n} e^{\lambda_1 n} \) and \( f_2(s) = \sum_{n=1}^{\infty} a_{2,n} e^{\lambda_2 n} \) be integral functions of orders \( \rho_1 (0<\rho_1<\infty) \), \( \rho_2 (0<\rho_2<\infty) \), then the function \( f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s} \), where (i) \( \lambda_{1,n} \sim \lambda_{2,n} \sim \lambda_n \) and (ii) \( |a_n| \sim |a_{1,n}|/|a_{2,n}| \), is an integral function of order \( \rho \) such that

\[
(2.1) \quad 1/\rho \geq 1/\rho_1 + 1/\rho_2.
\]

Further, if \( \gamma_1 (0<\gamma_1<\infty) \), \( \gamma_2 (0<\gamma_2<\infty) \) be the lower orders of \( f_1(s) \) and \( f_2(s) \) respectively, and (iii) \( \log \lambda_n \sim \log \lambda_{n+1} \), (iv) \( \log |a_{1,n}/a_{1,n+1}|/(\lambda_{1,n+1} - \lambda_{1,n}) \) and \( \log |a_{2,n}/a_{2,n+1}|/(\lambda_{2,n+1} - \lambda_{2,n}) \) nondecreasing functions of \( n \) for \( n>n_0 \), then

\[
(2.2) \quad 1/\lambda \leq 1/\gamma_1 + 1/\gamma_2,
\]

where \( \lambda \) is the lower order of \( f(s) \).

**Corollary.** If the conditions (i)-(iv) hold and both \( f_1(s) \) and \( f_2(s) \) be of linearly regular growth then so is the function \( f(s) \) and

\[
1/\rho = 1/\rho_1 + 1/\rho_2.
\]

**Proof.** Since \( \lambda_{1,n} \sim \lambda_{2,n} \sim \lambda_n \), it is evident that

\[
\limsup_{n \to \infty} \frac{\log n}{\lambda_n} = \limsup_{n \to \infty} \frac{\log n}{\lambda_{1,n}} = \limsup_{n \to \infty} \frac{\log n}{\lambda_{2,n}} = 0,
\]

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since both $f_1(s)$ and $f_2(s)$ satisfy (1.1) by hypothesis. Further, since $f_1(s)$ and $f_2(s)$ are integral functions, bounded in $\sigma < x$ for any $x < \infty$, the series $\sum_{n=1}^{\infty} |a_{1,n}| e^{\lambda_1 n}$ and $\sum_{n=1}^{\infty} |a_{2,n}| e^{\lambda_2 n}$ are convergent for every $\sigma$, and as $\lambda_n \sim \lambda_{1,n} \sim \lambda_{2,n}$ we shall have

$$\sum_{n=1}^{\infty} |a_{1,n}| e^{\lambda_n} < \infty$$

and

$$\sum_{n=1}^{\infty} |a_{2,n}| e^{\lambda_n} < \infty$$

for every $\sigma$. Hence, it follows that $\lim_{n \to \infty} |a_{1,n}| = 0$ and $\sum_{n=1}^{\infty} |a_{1,n}| e^{\lambda_n} < \infty$ and, since $|a_n| \sim |a_{1,n}| |a_{2,n}|$, we have $\sum_{n=1}^{\infty} |a_n| e^{\lambda_n} < \infty$ for every $\sigma$. Hence, $f(s)$ is an integral function.

Again, using (1.3) for $f_1(s)$ and $f_2(s)$, we get

$$\lim_{n \to \infty} \frac{-\log |a_{1,n}|}{\lambda_n \log \lambda_n} = 1/\rho_1,$$

since $\lambda_n \sim \lambda_{1,n}$ and

$$\lim_{n \to \infty} \frac{-\log |a_{2,n}|}{\lambda_n \log \lambda_n} = 1/\rho_2,$$

since $\lambda_n \sim \lambda_{2,n}$. Therefore, for any $\epsilon > 0$, we have for sufficiently large $n$,

$$\frac{-\log |a_{1,n}|}{\lambda_n \log \lambda_n} > 1/\rho_1 - \epsilon/2,$$

and

$$\frac{-\log |a_{2,n}|}{\lambda_n \log \lambda_n} > 1/\rho_2 - \epsilon/2.$$
Again, if \( \log \lambda_n \sim \log \lambda_{n+1} \), we have \( \log \lambda_{1_n} \sim \log \lambda_{1_{n+1}} \) and \( \log \lambda_{2_n} \sim \log \lambda_{2_{n+1}} \) in view of (i). Hence, if (iv) also holds, (1.6) of Theorem A gives

\[
\limsup_{n \to \infty} \frac{-\log |a_{1,n}|}{\lambda_n \log \lambda_n} = \frac{1}{\gamma_1},
\]

for the function \( f_1(s) \). Therefore, for any \( \epsilon > 0 \) and \( n > n_1 \geq n_0 \), we have

\[
(2.3) \quad \frac{-\log |a_{1,n}|}{\lambda_n \log \lambda_n} < \frac{1}{\gamma_1} + \frac{\epsilon}{2}.
\]

Similarly, for \( f_2(s) \) we have for \( n > n_2 \geq n_0 \)

\[
(2.4) \quad \frac{-\log |a_{2,n}|}{\lambda_n \log \lambda_n} < \frac{1}{\gamma_2} + \frac{\epsilon}{2}.
\]

Combining (2.3) and (2.4), we get

\[
\frac{-\log |a_{1,n}|}{\lambda_n \log \lambda_n} + \frac{-\log |a_{2,n}|}{\lambda_n \log \lambda_n} < \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \epsilon
\]

for sufficiently large \( n \). As \( |a_n| \sim |a_{1,n}| |a_{2,n}| \), this gives

\[
\limsup_{n \to \infty} \frac{-\log |a_n|}{\lambda_n \log \lambda_n} = \limsup_{n \to \infty} \frac{-\log |a_{1,n}|}{\lambda_n \log \lambda_n} + \frac{-\log |a_{2,n}|}{\lambda_n \log \lambda_n} \leq \frac{1}{\gamma_1} + \frac{1}{\gamma_2}.
\]

Or,

\[
1/\lambda \leq 1/\gamma_1 + 1/\gamma_2.
\]

The corollary follows from (2.1) and (2.2) since for functions of linearly regular growth, \( \rho_1 = \gamma_1, \rho_2 = \gamma_2 \) and so

\[
1/\lambda \leq 1/\gamma_1 + 1/\gamma_2 = 1/\rho_1 + 1/\rho_2 \leq 1/\rho.
\]

But \( \rho \geq \lambda \) and so \( 1/\lambda = 1/\rho_1 + 1/\rho_2 = 1/\rho \), which gives \( \rho = \lambda \). Hence, \( f(s) \) is also of linearly regular growth.

**Theorem 2.** If \( f_1(s) = \sum_{n=1}^{\infty} a_{1,n}e^{\lambda_1 n} \) and \( f_2(s) = \sum_{n=1}^{\infty} a_{2,n}e^{\lambda_2 n} \) be integral functions of linearly regular growth and such that (i) \( \lambda_{1,t} \sim \lambda_{2,t} \sim \lambda_t \), (ii) \( \log \lambda_n \sim \log \lambda_{n+1} \) and (iii) \( \log |a_{1,n}/a_{1,n+1}|/(\lambda_{1,n+1} - \lambda_{1,n}) \), \( \log |a_{2,n}/a_{2,n+1}|/(\lambda_{2,n+1} - \lambda_{2,n}) \) nondecreasing functions of \( n \) for \( n > n_0 \), then \( f_1(s), f_2(s) \) will be of the same finite order \( \rho \), if and only if

\[
(2.5) \quad \log \left\{ \frac{|a_{1,n}|}{|a_{2,n}|} \right\} = o(\lambda_n \log \lambda_n)
\]

as \( n \to \infty \).  

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Proof. Since \( f_1(s) \) satisfies the conditions of Theorem A and is also of regular growth, from (1.3) and (1.6), we have

\[
\limsup_{n \to \infty} \frac{\lambda_{1,n} \log \lambda_{1,n}}{\log |a_{1,n}|^{-1}} = \rho = \lambda = \liminf_{n \to \infty} \frac{\lambda_{1,n} \log \lambda_{1,n}}{\log |a_{1,n}|^{-1}}.
\]

Further, since \( \lambda_{1,n} \sim \lambda_n \), this gives

\[
(2.6) \quad \lim_{n \to \infty} \frac{-\log |a_{1,n}|}{\log \lambda_n} = 1/\rho.
\]

Similarly, for the function \( f_2(s) \), if it is of order \( \rho \), we have

\[
(2.7) \quad \lim_{n \to \infty} \frac{-\log |a_{2,n}|}{\log \lambda_n} = 1/\rho.
\]

Thus, (2.5) follows on subtracting (2.6) from (2.7).

Again, if \( \rho_1, \rho_2 \) be the orders of \( f_1(s), f_2(s) \), we have

\[
1/\rho_2 - 1/\rho_1 = \lim_{n \to \infty} \frac{\log \{|a_{1,n}| / |a_{2,n}|\}}{\log \lambda_n} = 0,
\]

if (2.5) holds. Hence, \( \rho_2 = \rho_1 \).

3. We shall now derive a result involving the ranks of the maximum terms in the integral function \( f(s) \) and its \( n \)th derivative \( f^{(n)}(s) \).

First we shall prove a lemma.

Lemma. If \( f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s} \) be an integral function of linear order \( \rho \) and lower linear order \( \lambda, \nu(\sigma, f), \nu(\sigma, f^{(n)}) \) denote the ranks of the maximum terms \( \mu(\sigma, f) \) and \( \mu(\sigma, f^{(n)}) \), for \( \Re(s) = \sigma \), in the series for \( f(s) \) and its \( n \)th derivative \( f^{(n)}(s) \), respectively, then

\[
(3.1) \quad \lim_{\sigma \to \infty} \sup_{\sigma_0} \left[ \frac{-}{\int_{\sigma}^{\sigma_0} X(t, n) dt} \right] = \frac{np}{n\lambda},
\]

where \( X(\sigma, n) = \lambda_{n(\sigma, f^{(n)})} - \lambda_{n(\sigma, f)} \), \( n = 1, 2, \ldots \).

Proof. Using (1.2) for \( f(s) \) and \( f^{(n)}(s) \), we have

\[
\log \mu(\sigma, f) = \int_{\sigma_0}^{\sigma} \lambda_{n(\sigma, f)} dt + O(1)
\]

and

\[
\log \mu(\sigma, f^{(n)}) = \int_{\sigma_0}^{\sigma} \lambda_{n(\sigma, f^{(n)})} dt + O(1).
\]

Therefore,
\[ \log \frac{\mu(\sigma, f^{(n)})}{\mu(\sigma, f)} = \int_{\sigma_0}^{\sigma} \left[ \lambda_{\nu(s, f^{(n)})} - \lambda_{\nu(s, f)} \right] dt + O(1). \]

Hence,

\[ \log \frac{\mu(\sigma, f^{(n)})}{\mu(\sigma, f)} = \frac{1}{\sigma} \int_{\sigma_0}^{\sigma} \chi(t, n) dt + o(1), \]

and (3.1) follows, since \([4, p. 89]\)

\[ \sup \left[ \log \frac{\mu(\sigma, f^{(n)})}{\mu(\sigma, f)} \right] = n\rho, \]

\[ \lim_{\sigma \to \infty} \inf \left[ \frac{\mu(\sigma, f)}{\sigma} \right] = n\lambda. \]

The following results are the direct consequences of the above lemma:

(i) For any \(\epsilon > 0\), we can find a \(\sigma_0\) such that

\[ (n\lambda - \epsilon)\sigma < \int_{\sigma_0}^{\sigma} \chi(t, n) dt < (n\rho + \epsilon)\sigma, \]

for \(\sigma > \sigma_0\).

(ii) If \(f(s)\) is of linearly regular growth \((\rho = \lambda)\),

\[ \int_{\sigma_0}^{\sigma} \chi(t, n) dt \sim n\rho\sigma \]

as \(\sigma \to \infty\).

**Theorem 3.** If \(f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}\) is an integral function of linearly regular growth and \(\lim_{s \to \infty} \chi(\sigma, n)\) exists then \(f(s)\) is of finite order \(\rho\) such that

\[ \lim_{\sigma \to \infty} \chi(\sigma, n) = n\rho, \]

where \(\chi(\sigma, n) = \lambda_{\nu(s, f^{(n)})} - \lambda_{\nu(s, f)}\), for \(n = 1, 2, \cdots, \nu(\sigma, f)\) and \(\nu(\sigma, f^{(n)})\) being the ranks of the maximum terms in \(f(s)\) and its \(n\)th derivative \(f^{(n)}(s)\) respectively.

**Proof.** We have

\[ \mu(\sigma, f) = |a_{\nu(s, f)}| \exp(\sigma \lambda_{\nu(s, f)}) \geq |a_{\nu(s, f^{(n)})}| \exp(\sigma \lambda_{\nu(s, f^{(n)})}) = \mu(\sigma, f^{(1)})/\lambda_{\nu(s, f^{(1)})}, \]

since \(\mu(\sigma, f^{(1)}) = \lambda_{\nu(s, f^{(1)})} |a_{\nu(s, f^{(1)})}| \exp(\sigma \lambda_{\nu(s, f^{(1)})})\). Also, \(\mu(\sigma, f^{(1)}) \geq \lambda_{\nu(s, f)} |a_{\nu(s, f)}| \exp(\sigma \lambda_{\nu(s, f)}) = \lambda_{\nu(s, f)} \mu(\sigma, f)\). Hence,
\[ \lambda_{\sigma(f,s)} \leq \frac{\mu(\sigma, f^{(1)})}{\mu(\sigma, f)} \leq \lambda_{\sigma(f^{(s)})}. \]

Using the above inequalities for the functions \( f^{(1)}(s), f^{(2)}(s), \ldots, \) etc., we get

\[ \lambda_{\sigma(f,s)} \leq \lambda_{\sigma(f^{(s)})} \leq \lambda_{\sigma(f^{(3)})} \leq \cdots \]

so that \( \chi(\sigma, n) = \lambda_{\sigma(f^{(s)})} - \lambda_{\sigma(f,s)} \geq 0, \) for \( n = 1, 2, \ldots. \) Now, if \( \lim_{s \to \infty} \chi(\sigma, n) = l, \) say, we have for any \( \epsilon > 0 \) and \( \sigma > \sigma_0, \)

\[ l - \epsilon < \chi(\sigma, n) < l + \epsilon. \]

Hence,

\[ \frac{(l - \epsilon)}{\sigma} \int_{s_0}^{s} dt < \frac{1}{\sigma} \int_{s_0}^{s} \chi(t, n) dt < \frac{(l + \epsilon)}{\sigma} \int_{s_0}^{s} dt, \]

or,

\[ (l - \epsilon)(1 - o(1)) < \frac{1}{\sigma} \int_{s_0}^{s} \chi(t, n) dt < (l + \epsilon)(1 - o(1)). \]

Therefore, on proceeding to limits, we get

\[ \lim_{\sigma \to \infty} \left[ \frac{1}{\sigma} \int_{s_0}^{s} \chi(t, n) dt \right] = l = np, \]

in view of (3.3).

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References


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