ON THE ORDER OF INTEGRAL FUNCTIONS DEFINED
BY DIRICHLET SERIES

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1. Consider the Dirichlet series

\[ f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n} \]

where \( \lambda_{n+1} > \lambda_n, \lambda_1 \geq 0, \lim_{n \to \infty} \lambda_n = \infty, s = \sigma + it \) and

\[ \limsup_{n \to \infty} \frac{\log n}{\lambda_n} = 0. \]  

Let \( \sigma_c \) and \( \sigma_a \) be the abscissa of convergence and the abscissa of absolute convergence, respectively, of \( f(s) \). If \( \sigma_c = \sigma_a = \infty \), \( f(s) \) is an integral function. We shall suppose throughout that (1.1) holds and that \( \sigma_c = \sigma_a = \infty \).

It is known that [1, p. 67]

\[ \log \mu(\sigma) = \log \mu(\sigma_0) + \int_{\sigma_0}^{\sigma} \nu(t) dt, \]  

where \( \nu(\sigma) \) gives the rank of the maximum term

\[ \mu(\sigma) = \max_{n \geq 1} \{ |a_n| e^{\lambda_n} \} \]

and

\[ \mu(\sigma) \leq M(\sigma), \]

where \( M(\sigma) = \max_{-\infty < t < \infty} |f(\sigma + it)| \). Also, if \( \rho \) be the Ritt-order [2 p. 78] of \( f(s) \) then [1, p. 69]

\[ \limsup_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}} = \rho = \limsup_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\sigma}. \]

A corresponding result for the lower order \( \lambda \) does not always hold. In fact, it has been shown that, if \( \log \lambda_n \sim \log \lambda_{n+1} \) then [3, p. 97]

\[ \lambda \geq \liminf_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}}, \]

where

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702
THE ORDER OF INTEGRAL FUNCTIONS

\[
\liminf_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\sigma} = \lambda \quad (0 \leq \lambda \leq \infty).
\]

But, if \( \log |a_n/a_{n+1}|/(\lambda_{n+1} - \lambda_n) \) is a nondecreasing function of \( n \) for \( n > n_0 \), then [3, p. 97]

\[
(1.5) \quad \lambda \leq \liminf_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}}.
\]

(1.4) and (1.5) lead to the following:

**Theorem A.** If \( f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n} \) is an integral function such that (i) \( \log \lambda_n \sim \log \lambda_{n+1} \), (ii) \( \log |a_n/a_{n+1}|/(\lambda_{n+1} - \lambda_n) \) form a nondecreasing function of \( n \) for \( n > n_0 \), then

\[
(1.6) \quad \liminf_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}} = \lambda \quad (0 \leq \lambda \leq \infty).
\]

The integral function \( f(s) \) is said to be of linearly regular growth if \( \rho = \lambda \). In this paper we derive a few properties of integral functions defined by Dirichlet series in relation to order and the ranks of their maximum terms.

2. **Theorem 1.** If \( f_1(s) = \sum_{n=1}^{\infty} a_{1,n} e^{\lambda_{1,n}} \) and \( f_2(s) = \sum_{n=1}^{\infty} a_{2,n} e^{\lambda_{2,n}} \) be integral functions of orders \( \rho_1 \) (\( 0 < \rho_1 < \infty \)), \( \rho_2 \) (\( 0 < \rho_2 < \infty \)), then the function \( f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n} \), where (i) \( \lambda_{1,n} \sim \lambda_{2,n} \sim \lambda_n \) and (ii) \( |a_n| \sim |a_{1,n}| |a_{2,n}| \), is an integral function of order \( \rho \) such that

\[
(2.1) \quad \frac{1}{\rho} \geq \frac{1}{\rho_1} + \frac{1}{\rho_2}.
\]

Further, if \( \gamma_1 \) (\( 0 < \gamma_1 < \infty \)), \( \gamma_2 \) (\( 0 < \gamma_2 < \infty \)) be the lower orders of \( f_1(s) \) and \( f_2(s) \) respectively, and (iii) \( \log \lambda_n \sim \log \lambda_{n+1} \), (iv) \( \log |a_{1,n}/a_{1,n+1}|/(\lambda_{1,n+1} - \lambda_{1,n}) \) and \( \log |a_{2,n}/a_{2,n+1}|/(\lambda_{2,n+1} - \lambda_{2,n}) \) nondecreasing functions of \( n \) for \( n > n_0 \), then

\[
(2.2) \quad \frac{1}{\lambda} \leq \frac{1}{\gamma_1} + \frac{1}{\gamma_2},
\]

where \( \lambda \) is the lower order of \( f(s) \).

**Corollary.** If the conditions (i)-(iv) hold and both \( f_1(s) \) and \( f_2(s) \) be of linearly regular growth then so is the function \( f(s) \) and

\[
1/\rho = 1/\rho_1 + 1/\rho_2.
\]

**Proof.** Since \( \lambda_{1,n} \sim \lambda_{2,n} \sim \lambda_n \), it is evident that

\[
\limsup_{n \to \infty} \frac{\log n}{\lambda_n} = \limsup_{n \to \infty} \frac{\log n}{\lambda_{1,n}} = \limsup_{n \to \infty} \frac{\log n}{\lambda_{2,n}} = 0,
\]

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since both \( f_1(s) \) and \( f_2(s) \) satisfy (1.1) by hypothesis. Further, since \( f_1(s) \) and \( f_2(s) \) are integral functions, bounded in \( \sigma < x \) for any \( x < \infty \), the series \( \sum_{n=1}^{\infty} |a_{1,n}| e^{\sigma \lambda_1,n} \) and \( \sum_{n=1}^{\infty} |a_{2,n}| e^{\sigma \lambda_2,n} \) are convergent for every \( \sigma \), and as \( \lambda_n \sim \lambda_{1,n} \sim \lambda_{2,n} \) we shall have

\[
\sum_{n=1}^{\infty} |a_{1,n}| e^{\sigma \lambda_n} < \infty
\]

and

\[
\sum_{n=1}^{\infty} |a_{2,n}| e^{\sigma \lambda_n} < \infty
\]

for every \( \sigma \). Hence, it follows that \( \lim_{n \to \infty} |a_{1,n}| = 0 \) and \( \sum_{n=1}^{\infty} \frac{|a_{1,n}|}{e^{\sigma \lambda_n}} < \infty \) and, since \( |a_n| \sim |a_{1,n}| / |a_{2,n}| \), we have \( \sum_{n=1}^{\infty} |a_n| e^{\sigma \lambda_n} < \infty \) for every \( \sigma \). Hence, \( f(s) \) is an integral function.

Again, using (1.3) for \( f_1(s) \) and \( f_2(s) \), we get

\[
\liminf_{n \to \infty} \frac{-\log |a_{1,n}|}{\lambda_n \log \lambda_n} = 1/p_1,
\]

since \( \lambda_n \sim \lambda_{1,n} \) and

\[
\liminf_{n \to \infty} \frac{-\log |a_{2,n}|}{\lambda_n \log \lambda_n} = 1/p_2,
\]

since \( \lambda_n \sim \lambda_{2,n} \). Therefore, for any \( \epsilon > 0 \), we have for sufficiently large \( n \),

\[
\frac{-\log |a_{1,n}|}{\lambda_n \log \lambda_n} > 1/p_1 - \epsilon/2,
\]

and

\[
\frac{-\log |a_{2,n}|}{\lambda_n \log \lambda_n} > 1/p_2 - \epsilon/2.
\]

Or,

\[
\frac{-\log |a_{1,n}| / |a_{2,n}|}{\lambda_n \log \lambda_n} > 1/p_1 + 1/p_2 - \epsilon.
\]

And, since \( |a_n| \sim |a_{1,n}| / |a_{2,n}| \), we get,

\[
\liminf_{n \to \infty} \frac{-\log |a_n| / |a_{1,n}| / |a_{2,n}|}{\lambda_n \log \lambda_n} = \liminf_{n \to \infty} \frac{-\log |a_{1,n}| / |a_{2,n}|}{\lambda_n \log \lambda_n} \equiv 1/p_1 + 1/p_2.
\]

Or,
Again, if \( \log \lambda \sim \log \lambda_{n+1} \), we have \( \log \lambda_{1,n} \sim \log \lambda_{1,n+1} \) and \( \log \lambda_{2,n} \sim \log \lambda_{2,n+1} \) in view of (i). Hence, if (iv) also holds, (1.6) of Theorem A gives

\[
\limsup_{n \to \infty} \frac{-\log |a_{1,n}|}{\lambda_n \log \lambda_n} = 1/\gamma_1,
\]

for the function \( f_1(s) \). Therefore, for any \( \epsilon > 0 \) and \( n > n_1 \geq n_0 \), we have

\[
(2.3) \quad \frac{-\log |a_{1,n}|}{\lambda_n \log \lambda_n} < 1/\gamma_1 + \epsilon/2.
\]

Similarly, for \( f_2(s) \) we have for \( n > n_2 \geq n_0 \)

\[
(2.4) \quad \frac{-\log |a_{2,n}|}{\lambda_n \log \lambda_n} < 1/\gamma_2 + \epsilon/2.
\]

Combining (2.3) and (2.4), we get

\[
\limsup_{n \to \infty} \frac{-\log |a_{1,n}|}{\lambda_n \log \lambda_n} = \limsup_{n \to \infty} \frac{-\log |a_{1,n}|}{\lambda_n \log \lambda_n} \leq 1/\gamma_1 + 1/\gamma_2.
\]

Or,

\[
\frac{1}{\lambda} \leq 1/\gamma_1 + 1/\gamma_2.
\]

The corollary follows from (2.1) and (2.2) since for functions of linearly regular growth, \( \rho_1 = \gamma_1, \rho_2 = \gamma_2 \) and so

\[
\frac{1}{\lambda} \leq 1/\gamma_1 + 1/\gamma_2 = 1/\rho_1 + 1/\rho_2 \leq 1/\rho.
\]

But \( \rho \geq \lambda \) and so \( 1/\lambda = 1/\rho_1 + 1/\rho_2 = 1/\rho \), which gives \( \rho = \lambda \). Hence, \( f(s) \) is also of linearly regular growth.

**Theorem 2.** If \( f_1(s) = \sum_{n=1}^{\infty} a_{1,n}e^{\lambda_1,n} \) and \( f_2(s) = \sum_{n=1}^{\infty} a_{2,n}e^{\lambda_2,n} \) be integral functions of linearly regular growth and such that (i) \( \lambda_{1,n} \sim \lambda_{2,n} \sim \lambda_n \), (ii) \( \log \lambda_n \sim \log \lambda_{n+1} \) and (iii) \( \log |a_{1,n}/a_{1,n+1}|/(\lambda_{1,n+1} - \lambda_{1,n}) \), \( \log |a_{2,n}/a_{2,n+1}|/(\lambda_{2,n+1} - \lambda_{2,n}) \) nondecreasing functions of \( n \) for \( n > n_0 \), then \( f_1(s), f_2(s) \) will be of the same finite order \( \rho \), if and only if

\[
(2.5) \quad \log \{ |a_{1,n}|/|a_{2,n}| \} = o(\lambda_n \log \lambda_n)
\]

as \( n \to \infty \).
Proof. Since $f_1(s)$ satisfies the conditions of Theorem A and is also of regular growth, from (1.3) and (1.6), we have

$$\limsup_{n \to \infty} \frac{\lambda_{1,n}}{\log |a_{1,n}|^{-1}} = \rho = \lambda = \liminf_{n \to \infty} \frac{\lambda_{1,n}}{\log |a_{1,n}|^{-1}}.$$  

Further, since $\lambda_{1,n} \sim \lambda_n$, this gives

$$\lim_{n \to \infty} -\frac{\log |a_{1,n}|}{\lambda_n \log \lambda_n} = 1/\rho.$$  

Similarly, for the function $f_2(s)$, if it is of order $\rho$, we have

$$\lim_{n \to \infty} -\frac{\log |a_{2,n}|}{\lambda_n \log \lambda_n} = 1/\rho.$$  

Thus, (2.5) follows on subtracting (2.6) from (2.7).

Again, if $\rho_1, \rho_2$ be the orders of $f_1(s), f_2(s)$, we have

$$\frac{1}{\rho_2} - 1/\rho_1 = \lim_{n \to \infty} \frac{\log |a_{1,n}| / |a_{2,n}|}{\lambda_n \log \lambda_n} = 0,$$

if (2.5) holds. Hence, $\rho_2 = \rho_1$.

3. We shall now derive a result involving the ranks of the maximum terms in the integral function $f(s)$ and its $n$th derivative $f^{(n)}(s)$. First we shall prove a lemma.

Lemma. If $f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}$ be an integral function of linear order $\rho$ and lower linear order $\lambda, v(\sigma, f), \nu(\sigma, f^{(n)})$ denote the ranks of the maximum terms $\mu(\sigma, f)$ and $\mu(\sigma, f^{(n)})$, for $\text{Re}(s) = \sigma$, in the series for $f(s)$ and its $n$th derivative $f^{(n)}(s)$, respectively, then

$$\lim_{\sigma \to \infty} \sup_{\sigma_0} \left[ -\frac{1}{\sigma} \int_{\sigma_0}^{\sigma} \chi(t, n) \, dt \right] = \frac{np}{n\lambda},$$

where $\chi(\sigma, n) = \lambda_{v(\sigma, f^{(n)})} - \lambda_{v(\sigma, f)}$, $n = 1, 2, \ldots$.

Proof. Using (1.2) for $f(s)$ and $f^{(n)}(s)$, we have

$$\log \mu(\sigma, f) = \int_{\sigma_0}^{\sigma} \lambda_{v(\sigma, f)} \, dt + O(1)$$

and

$$\log \mu(\sigma, f^{(n)}) = \int_{\sigma_0}^{\sigma} \lambda_{v(\sigma, f^{(n)})} \, dt + O(1).$$

Therefore,
\[
\log \frac{\mu(\sigma, f^{(n)})}{\mu(\sigma, f)} = \int_{e_0}^{\sigma} [\lambda_{\nu(t, f^{(n)})} - \lambda_{\nu(t, f)}] dt + O(1).
\]
Hence,
\[
\log \frac{\mu(\sigma, f^{(n)})}{\mu(\sigma, f)} = \frac{1}{\sigma} \int_{e_0}^{\sigma} \chi(t, n) dt + o(1),
\]
and (3.1) follows, since [4, p. 89]
\[
\sup \left[ \frac{\log \mu(\sigma, f^{(n)})}{\mu(\sigma, f)} \right] \rightarrow n\rho,
\]
\[
\lim_{\sigma \rightarrow \infty} \inf \left[ \frac{\log \mu(\sigma, f^{(n)})}{\mu(\sigma, f)} \right] = n\lambda.
\]

The following results are the direct consequences of the above lemma:

(i) For any \(\epsilon > 0\), we can find a \(\sigma_0\) such that
\[
(n\lambda - \epsilon)\sigma < \int_{e_0}^{\sigma} \chi(t, n) dt < (n\rho + \epsilon)\sigma,
\]
for \(\sigma > \sigma_0\).

(ii) If \(f(s)\) is of linearly regular growth \((\rho = \lambda)\),
\[
(3.3) \quad \int_{e_0}^{\sigma} \chi(t, n) dt \sim n\rho \sigma
\]
as \(\sigma \rightarrow \infty\).

**Theorem 3.** If \(f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}\) is an integral function of linearly regular growth and \(\lim_{\sigma \rightarrow \infty} \chi(\sigma, n)\) exists then \(f(s)\) is of finite order \(\rho\) such that
\[
(3.4) \quad \lim_{\sigma \rightarrow \infty} \chi(\sigma, n) = n\rho,
\]
where \(\chi(\sigma, n) = \lambda_{\nu(s,f^{(n)})} - \lambda_{\nu(s,f)},\) for \(n = 1, 2, \ldots, \nu(s, f)\) and \(\nu(s, f^{(n)})\)
being the ranks of the maximum terms in \(f(s)\) and its \(n\)th derivative \(f^{(n)}(s)\)
respectively.

**Proof.** We have
\[
\mu(\sigma, f) = |a_{\nu(s,f)}| \exp(\sigma\lambda_{\nu(s,f)})
\geq |a_{\nu(s,f^{(n)})}| \exp(\sigma\lambda_{\nu(s,f^{(n)})})
= \mu(\sigma, f^{(1)}) / \lambda_{\nu(s,f^{(1)})},
\]
since \(\mu(\sigma, f^{(1)}) = \lambda_{\nu(s,f^{(1)})} |a_{\nu(s,f^{(1)})}| \exp(\sigma\lambda_{\nu(s,f^{(1)})})\). Also, \(\mu(\sigma, f^{(1)}) \geq \lambda_{\nu(s,f)} |a_{\nu(s,f)}| \exp(\sigma\lambda_{\nu(s,f)}) = \lambda_{\nu(s,f)} \mu(\sigma, f).\) Hence,
Using the above inequalities for the functions \( f^{(1)}(s), f^{(2)}(s), \ldots \), etc., we get
\[
\lambda_{\sigma}(f^{(1)}) \leq \lambda_{\sigma}(f^{(2)}) \leq \lambda_{\sigma}(f^{(3)}) \leq \cdots
\]
so that \( \chi(\sigma, n) = \lambda_{\sigma}(f^{(n)}) - \lambda_{\sigma}(f^{(n-1)}) \geq 0 \), for \( n = 1, 2, \ldots \). Now, if \( \lim_{n \to \infty} \chi(\sigma, n) = l \), say, we have for any \( \epsilon > 0 \) and \( \sigma > \sigma_0 \),
\[
l - \epsilon < \chi(\sigma, n) < l + \epsilon.
\]
Hence,
\[
\frac{(l - \epsilon)}{\sigma} \int_{\sigma_0}^{\sigma} dt < \frac{1}{\sigma} \int_{\sigma_0}^{\sigma} \chi(t, n) dt < \frac{(l + \epsilon)}{\sigma} \int_{\sigma_0}^{\sigma} dt,
\]
or,
\[
(l - \epsilon)(1 - o(1)) < \frac{1}{\sigma} \int_{\sigma_0}^{\sigma} \chi(t, n) dt < (l + \epsilon)(1 - o(1)).
\]
Therefore, on proceeding to limits, we get
\[
\lim_{\sigma \to \infty} \left[ \frac{1}{\sigma} \int_{\sigma_0}^{\sigma} \chi(t, n) dt \right] = l = np,
\]
in view of (3.3).

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References


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