I. Introduction. Let \( K \) denote the class of functions

\[
f(z) = \sum_{r=1}^{\infty} a_r z^r
\]

which are regular in the unit circle and map it onto a schlicht convex domain. In a recent paper [1] Pólya and Schoenberg have shown that in order for \( f(z) \in K \) it is necessary and sufficient that each of the functions

\[
V_n(z; f) = \frac{1}{C_{2n,n}} \sum_{r=1}^{n} C_{2n,n+r} a_r z^r \quad (n = 1, 2, \cdots)
\]

be \( K \).

We will use the notation

\[
f(z) \subseteq g(z)
\]

("\( f(z) \) is subordinate to \( g(z) \)") to mean that \( f(z), g(z) \) are both regular in \( |z| < 1 \), that \( g(z) \) is univalent there, and that every value taken by \( f(z) \) in \( |z| < 1 \) is also taken by \( g(z) \) (see [2; 3]). It was shown in [1] that

\[
V_n(z; f) \subseteq f(z) \quad (n = 1, 2, \cdots)
\]

for every \( f(z) \in K \), and it was pointed out that even

\[
V_1(z; f) \subseteq V_2(z; f) \subseteq \cdots \subseteq f(z)
\]

is likely, though this was not verified except for

\[
f_0(z) = z(1 - z)^{-1}.
\]

In the following paragraphs we will show how consideration of the problem (5) leads, in a natural way, to the question of characterizing certain kinds of factor sequences (see [6]), and although we cannot decide the truth or falsity of (5), a closely related question will be completely settled (Theorem 2, infra).
II. Subordinating factor sequences. An infinite sequence \( \{ b_r \} \) of complex numbers will be called a subordinating factor sequence if whenever

\[
f(z) = \sum_{r=1}^{\infty} a_r z^r \in K
\]

we have

\[
\sum_{r=1}^{\infty} a_r b_r z^r \subseteq f(z).
\]

A finite sequence \( \{ b_r \} \) will be called a subordinating factor sequence if (7) implies (8) whenever \( a_{n+1} = a_{n+2} = \cdots = 0 \). The class of such infinite sequences we denote by \( \mathcal{F} \), and that of sequences of length \( n \) by \( \mathcal{F}_n \).

**Theorem 1.** The proposition

\[
\left\{ 1 - \frac{\nu^2}{n^2} \right\}^n \in \mathcal{F}_n
\]

implies (5).

**Proof.** This is immediate from the easily established identity

\[
\left( 1 + \frac{z}{n} \frac{d}{dz} \right) \left( 1 - \frac{z}{n} \frac{d}{dz} \right) V_n(z; f) = V_{n-1}(z; f) \quad (n = 2, 3, \cdots)
\]

and the definition of \( \mathcal{F}_n \).

We do not know how to characterize sequences of \( \mathcal{F}_n \). The following result, however, completely describes the class \( \mathcal{F} \).

**Theorem 2.** The following three properties of a sequence of complex numbers are equivalent:

(I) \( \{ b_r \} \subseteq \mathcal{F} \);

(II) \( \Re \left\{ 1 + 2 \sum_{r=1}^{\infty} b_r z^r \right\} > 0 \) \quad (|z| < 1);

(III) \( b_r = \frac{1}{2\pi} \int_0^{2\pi} e^{i\nu \theta} d\psi(\theta) \quad (\nu = 0, 1, 2, \cdots; b_0 = 1; \psi(\theta) \uparrow) \).

**Proof.** The equivalence of (II) and (III) is classical. Now suppose (I) holds. Then
\[ \sum_{r=1}^{\infty} b_r z^r \subseteq \sum_{r=1}^{\infty} z^r = z(1 - z)^{-1}, \]

which is to say that,

\[ \text{Re} \left\{ \sum_{r=1}^{\infty} b_r z^r \right\} > -\frac{1}{2} \quad (|z| < 1), \]

which proves (II). Conversely, if (III) holds, let

(11) \[ f(z) = \sum_{r=1}^{\infty} a_r z^r \in K. \]

Then

\[ \sum_{r=1}^{\infty} a_r b_r z^r = \frac{1}{2\pi} \int_{0}^{2\pi} \sum_{r=1}^{\infty} a_r e^{ir\theta} e^{ir\phi} d\psi(\theta) \]

(12)

\[ = \frac{1}{2\pi} \int_{0}^{2\pi} f(re^{i(\theta+\phi)}) d\psi(\theta). \]

The left hand side is thus exhibited as the centroid of a nonnegative mass distribution of total mass one, on a convex curve, and therefore lies inside that curve, which was to be shown.

Several results, some well known, follow immediately from Theorem 2.

**Corollary 1.** If \( \{b_r\}_{r=1}^{\infty} \in \mathfrak{F}, \{c_r\}_{r=1}^{\infty} \in \mathfrak{F}, \) then \( \{b_r c_r\}_{r=1}^{\infty} \in \mathfrak{F}. \)

Since from (12), the result of applying a sequence of \( \mathfrak{F} \) to an arbitrary analytic function is a function which maps the unit circle into the convex hull of the original image, the result of applying these two sequences to \( f(z) \) in succession is clearly subordinate to \( f(z) \).

**Corollary 2.** If

\[ \text{Re} \left\{ 1 + 2 \sum_{r=1}^{\infty} a_r z^r \right\} > 0, \quad \text{Re} \left\{ 1 + 2 \sum_{r=1}^{\infty} b_r z^r \right\} > 0 \]

then

\[ \text{Re} \left\{ 1 + 2 \sum_{r=1}^{\infty} a_r b_r z^r \right\} > 0 \quad (|z| < 1). \]

This well-known result [5, VII, 43] is clear from Theorem 2 and Corollary 1.

**Corollary 3.** The image of the unit circle under the mapping
of $K$, contains the circle $|W| < 1/2$, the constant being sharp.

This result, due to Study [4] (compare [2, p. 223]; [1, p. 320]) is precisely the assertion that the sequence $1/2, 0, 0, \cdots$ belongs to $\mathcal{S}$, which is obvious from Theorem 2, (11). The sharpness is shown, as usual, by the example (6).

**Corollary 4.** Equation (4) is true.

Indeed, from (6) and (6') of [1] with $z = e^{i\theta}$, there follows

\[
\text{Re}\left\{1 + 2V_n\left(z; \frac{z}{1 - z}\right)\right\} = 1 + 2 \sum_{r=1}^{n} \frac{n!}{(n - r)!} \frac{n!}{(n + r)!} \cos \nu \theta = \frac{(n!)^2}{(2n)!} \left(\frac{2 \cos \frac{\theta}{2}\right)^{2n} \geq 0
\]

whence the sequence

\[
b_\nu = \begin{cases} 
C_{2n,n+\nu}/C_{2n,n} & (\nu = 1, 2, \cdots, n) \\
0 & (\nu \geq n + 1)
\end{cases}
\]

belongs to $\mathcal{S}$, which is exactly what (4) asserts (our proof is really identical with that in [1]).

**Corollary 5.** Let the functions $f(z) = \sum_{\nu} a_\nu z^\nu, g(z) = \sum_{\nu} b_\nu z^\nu$ belong to $K$, and map $|z| < 1$ onto domains $\mathcal{D}'$, $\mathcal{D}''$, respectively, both contained in $\text{Re } w > -1/2$. Then the function $\sum_{\nu} a_\nu b_\nu z^\nu$ maps $|z| < 1$ onto a domain $\mathcal{D} \subseteq \mathcal{D}' \cap \mathcal{D}''$.

This result, which is related to a conjecture of Pólya-Schoenberg on the Hadamard product of functions of $K$, follows by noting that $\{a_{\nu}\}_{\mathcal{S} \subseteq \mathcal{D} \subseteq \mathcal{D}''}, \{b_{\nu}\}_{\mathcal{S} \subseteq \mathcal{D} \subseteq \mathcal{D}''}$ which was to be shown.

Concerning the open question (9) we may now see that, in any event, the sequence $\{1 - \nu^2/n^2\}_{\nu}^\infty$ is not extendable to a sequence of $\mathcal{S}$ since that would require the positivity of the Toeplitz matrix

\[
T = \begin{bmatrix}
1 & 1 - 1/n^2 & 1 - 4/n^2 & \cdots \\
1 - 1/n^2 & 1 & 1 - 1/n^2 & \cdots \\
1 - 4/n^2 & 1 - 1/n^2 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]
whereas the $3 \times 3$ determinant in the upper left corner has the value $-8n^{-8} < 0$.

III. On the composition of convex maps. In [1] it was conjectured that if $f(z) = \sum a_nz^n \in K$ and $g(z) = \sum b_nz^n \in K$ then so does $h(z) = \sum a_nb_nz^n$. We state here a single proposition whose truth would imply both this conjecture and (5) at once. It is

**Proposition 1.** The coefficients of a convex function preserve subordination between convex functions. That is, if $\sum a_nz^n$, $\sum b_nz^n$, $\sum c_nz^n$ are all in $K$, and if

$$
\sum a_nz^n \subseteq \sum b_nz^n
$$

then

$$
\sum a_nz^n \subseteq \sum c_nz^n.
$$

Indeed, if this is true, then since (5) holds in the case (6) it holds in general. Further, by applying the sequence $\{b_n\}$ to the relations

$$
V_n(z; f) \subseteq f(z) \quad (n = 1, 2, \ldots)
$$

we would find that the means of the function $\sum a_nb_nz^n$ are subordinate to the function itself, and in view of a recent result of Robertson [7], it would follow that $\sum a_nb_nz^n \in K$.

**References**