ON THE SPECTRUM OF A CONTRACTION

M. SCHREIBER

1. Introduction. In this note we present several results on the spectrum of a contraction. The first is an extension to the approximate point spectrum of a result of Nagy and Foias, on the relation of the point spectrum of a contraction and that of its unitary dilation, which has several corollaries. The second is a simple solution to a problem in spectral mapping raised in [2]. Finally we have a result on the point spectrum of a class of contractions discussed in [3]. For the background on unitary dilations see [4] or [5].

2. Arbitrary contractions. In Theorem 1 of [6] it is shown that the set of eigenvalues of modulus 1 of a contraction \( A \) coincides with that of its unitary dilation \( U \). Less is true for the approximate point spectrum \( \Sigma_{ap} A \). (See [1] for the definition of \( \Sigma_{ap} \).)

Proposition. Let \( A \) be a contraction on a Hilbert space \( H \) and let \( U \) be a unitary dilation on a (larger) space \( K \). Then \( \mu = e^{ix} \in \Sigma_{ap} A \) if and only if \( \mu \in \Sigma_{ap} U \) with approximate eigenvectors in \( H \).

(Thus, if \( \mu \in \Sigma_{ap} U \), \( |\mu| = 1 \), but the approximate eigenvectors are not in \( H \), then \( \mu \notin \Sigma_{ap} A \).)

Proof. Let \( P \) be the projection of \( K \) onto \( H \). If there are unit vectors \( x_n \in H \) with \( \| Ux_n - \mu x_n \| \to 0 \) as \( n \to \infty \), then \( \| Ax_n - \mu x_n \| = \| Px_n - \mu x_n \| \leq \| Ux_n - \mu x_n \| \to 0 \) as \( n \to \infty \), so that \( \mu \in \Sigma_{ap} A \). For the converse, there is clearly no loss of generality in taking \( \mu = 1 \), and we suppose there are unit vectors \( x_n \in H \) such that

\[
\| Ax_n - x_n \| \leq 1/n, \quad n = 1, 2, \ldots,
\]

from which it follows that \( \| Ax_n \| \geq 1 - 1/n \). Again let \( P \) be the projection of \( K \) onto \( H \), and write \( H^\perp \) for the orthogonal complement of \( H \) in \( K \). Now \( Ux_n = u_n + v_n \), with \( u_n \in H \), \( v_n \in H^\perp \), and \( \| u_n \|^2 + \| v_n \|^2 = \| Ux_n \|^2 = \| x_n \|^2 = 1 \). Since \( u_n = PUx_n = Ax_n \), we have

Received by the editors August 3, 1960 and, in revised form, October 3, 1960.

1 Research for this paper was sponsored by the National Science Foundation Contract No. G5253.

2 We are grateful to the referee for simplifications of the arguments in this and the following paragraph.

3 By a unitary dilation of an operator \( A \) on \( H \) is meant a unitary operator \( U \) on a space \( K \supset H \) such that \( PUx = Ax \) for all \( x \in H \), where \( P \) is the projection of \( K \) onto \( H \). In [4; 5] a unique minimal such dilation is studied, but for present purposes minimality is irrelevant.

709
1 = \|Ax_n\|^2 + \|v_n\|^2 \geq 1 - \frac{2}{n} + \frac{1}{n^2} + \|v_n\|^2,

so that

\|v_n\|^2 \leq \frac{2}{n} - \frac{1}{n^2}.

The components in $H$ and $H^1$ of $Ux_n - x_n$, the vector whose norm is to be estimated, are $Ax_n - x_n$ and $v_n$, respectively, as is clear, and so

\|Ux_n - x_n\|^2 = \|Ax_n - x_n\|^2 + \|v_n\|^2 \leq \frac{1}{n^2} + \frac{2}{n} - \frac{1}{n^2} = \frac{2}{n},

by the displayed inequalities, whence $\|Ux_n - x_n\| \leq (2/n)^{1/2}$ and the proof is complete.

[Added in proof. Professor Sz.-Nagy has remarked in a private communication that the proposition may be proved very simply as follows. With notation as above, for $x \in H$ we have

\|Ux - x\|^2 = \|Ux\|^2 + \|x\|^2 - 2 \operatorname{Re}(Ux, x)

= 2\|x\|^2 - 2 \operatorname{Re}(Tx, x)

= 2 \operatorname{Re}(x, x - Tx) \leq 2\|x\|\|x - Tx\|,

and the conclusion follows at once.]

**Corollary 1.** The approximate eigenvalues of modulus 1 of $A^*$ are the complex conjugates of those of $A$.

**Proof.** If $\mu = e^{i\alpha} \in \Sigma_{ap}A$ then $\mu \in \Sigma_{ap}U$ with approximate eigenvectors in $H$, so that given $\epsilon > 0$ there exists a unit vector $x \in H$ with $\|Ux - \mu x\| < \epsilon$. Hence $\|U^*x - \bar{\mu} x\| < \epsilon$, trivially, and by the proposition again it follows that $\|A^*x - \bar{\mu} x\| < \epsilon$, as was to be shown.

The same result for the point spectrum is given in [5, p. 88].

**Corollary 2.** Near a gap in $\Sigma U$ there can be only residual spectrum of $A$.

**Proof.** By a gap in $\Sigma U$ is meant an open arc of the unit circle which lies in the complement of $\Sigma U$, and the assertion is that every such gap is contained in a planar open set disjoint from $\Sigma_{ap}A$. The proof is based on the closure of $\Sigma_{ap}A$. Suppose this for the moment. Let $G$ be a gap in $\Sigma U$ and $e^{i\alpha} \in G$. Then there must be an open circle $C_\alpha$ centered at $e^{i\alpha}$ with $C_\alpha \cap \Sigma_{ap}A = \emptyset$, else $e^{i\alpha}$ would be a limit point of $\Sigma_{ap}A$, hence in $\Sigma_{ap}A$, and therefore by the proposition a member of $\Sigma_{ap}U$, contrary to supposition. The open set required by the corollary is then $U x C_\alpha$. We complete the proof by showing that $\Sigma_{ap}A$ is closed,
for any bounded $A$. Let $\lambda_n \in \Sigma_{ap} A$, $\lambda_n \to \lambda$. If $\lambda \in \Sigma_{ap} A$ then there exists $\epsilon > 0$ such that $\| (A - \lambda I) x \| \geq \epsilon$ for all unit vectors $x$. Then $\| \lambda - \lambda_n \| = \| (A - \lambda I) x - (A - \lambda_n I) x \| \geq \epsilon - \| (A - \lambda_n I) x \| \geq \epsilon - \| \lambda - \lambda_n \|$ for all unit vectors $x$. In particular, if $\| \lambda_n - \lambda \| \leq \epsilon/2$ then $\| (A - \lambda_n I) x \| \geq \epsilon/2$ for all unit vectors, so that $\lambda_n \in \Sigma_{ap} A$, contrary to supposition.

A side condition such as the one employed in the proposition (that the approximate eigenvectors for $U$ be in $H$) is seen to be necessary by taking for $A$ any contraction with no spectrum on $\{ |z| = 1 \}$, whereas $\Sigma U \subseteq \{ |z| = 1 \}$ and $\Sigma U = \Sigma_{ap} U$ (see [1, p. 51]). In particular we know that for such $A$ the approximate eigenvectors of its dilation $U$ cannot be in $H$.

3. A spectral mapping problem. In [2] we studied the preservation of $\Sigma_{ap} A$ under general mappings and noted that in general it is not preserved in the reverse direction (that is, $\alpha \in f^{-1}(\beta)$ need not be an eigenvalue of $A$ when $\beta \in f(A)$). On the other hand it is trivially clear that if for all function $f$ the number $f(\mu)$ is an eigenvalue of $f(A)$ then $\mu$ is an eigenvalue of $A$. The problem is to find a nonvacuous condition sufficient for preservation of $\Sigma_{ap} A$ in the reverse direction.

Let $C_n(f)$ be the $n$th Taylor coefficient of $f$, and write $f_t$ for the function $f(t_0) = f(ts)$.

Proposition. Let $A$ be a contraction, and $f$ a fixed function analytic for $|z| < 1$. If $f_t(A)x = f_t(\mu)x$ for infinitely many (complex) $t$ converging inside the unit circle, then $A^m x = \mu^m x$, where $m$ is the least $n > 0$ such that $C_n(f) \neq 0$. Conversely, if $A^m x = \mu^m x$ and $C_k(f) = 0$ for $0 \leq k < m$ then $f_t(A)x = f_t(\mu)x$ for all $|t| < 1$.

Proof. By hypothesis $f(z) = \sum_0^\infty C_n(f) z^n$ converges for $|z| < 1$, so $f_t(z) = \sum_0^\infty C_n(f) t^n z^n$ has radius of convergence $r(t) > 1$ for $|t| < 1$. Since $\| A \| \leq 1$ the operator series $\sum_0^\infty C_n(f) A^n t^n$ converges in norm, for $|t| < 1$, to an operator which we define as $f_t(A)$, so that

$$(f_t(A)x, y) = \sum_0^\infty C_n(f)(A^n x, y) t^n = F(t), \quad |t| < 1,$$

is an analytic function of $t$, for each pair $x, y$ of vectors. (This definition of $f_t(A)$ agrees with that of [4], $(f_t(A)x, y) = \int f_t(e^{it}) d F(s)x, y)$, for by uniform convergence the integral is equal to

$$\sum_0^\infty C_n(f) t^n \int e^{isa} d F(s)x, y) = \sum C_n(f) t^n (A^n x, y),$$
and it is easy to see also that it agrees with the classical definition by
the Cauchy integral formula.) Similarly \( f_s(\mu)(x, y) = (f_s(\mu)x, y) \) may
be expanded in the series

\[
(f_s(\mu)x, y) = \sum_{0}^{\infty} C_n(f)(\mu^n x, y)t^n \equiv G(t), \quad |t| < 1.
\]

Now for the first assertion of the proposition we have by hypothesis
that \( F = G \) on an infinite set with limit point inside the circle. Since
\( F \) and \( G \) are clearly analytic for \( |t| < 1 \) we conclude that \( F(t) = G(t) \),
\( |t| < 1 \). This means that, for all \( n \geq 0 \) and all \( y \in H \),

\[
c_n(f)(\mu^n x, y) = c_n(f)(x, y),
\]

and the assertion now follows by cancellation of \( c_n(f) \). The second
assertion goes in the same spirit. The hypotheses involve \( F(t) = G(t) \)
for \( |t| < 1 \) and therefore \( f_s(A)x, y = (f_s(\mu)x, y) \) for all \( y \) and \( |t| < 1 \),
which yields the conclusion.

4. A contraction \( A \) is absolutely continuous if there exists a func-
tion \( K(t, x, y) \in L_1(0, 2\pi) \) for every pair of vectors \( x, y \), such that

\[
(A^n x, y) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{int} K(t, x, y)dt
\]

for all \( n = 0, \pm 1, \cdots \) (here \( A^{-n} = A^{*n} \) (see [3]). This is a smooth-
ness condition which reflects itself in the spectrum of \( A \) as follows:

**Proposition.** An absolutely continuous contraction has no eigen-
values of modulus \( 1 \).

**Proof.** Let \( A \) be absolutely continuous. The representation above
for \( A \) in terms of \( K \) amounts to the assertion that \( K \) has the Fourier
expansion

\[
K(t, x, y) \sim \sum_{-\infty}^{\infty} e^{-int}(A^n x, y).
\]

Now suppose that \( Ax = e^{i\beta}x \) for some unit vector \( x \) and \( 0 \leq \beta \leq 2\pi \). It
then follows from [5, p. 88] that \( A^n x = e^{in\beta}x \) for \( n = 0, \pm 1, \pm 2, \cdots \).
Hence the Fourier expansion for \( K(\cdot, x, x) \) reduces to

\[
K(t, x, x) \sim \sum_{-\infty}^{\infty} e^{in\beta} e^{-int}.
\]

But \( K(\cdot, x, x) \in L_1 \), so that its Fourier coefficients \( e^{in\beta} \) must tend to 0.
This contradiction completes the proof.
A SUBSTITUTE FOR LEBESGUE'S BOUNDED CONVERGENCE THEOREM

I. NAMIOKA

1. Lebesgue's bounded convergence theorem has become a powerful tool in the theory of linear topological spaces, and recently, for a treatment of weak convergence of sequences or for a proof of Krein's theorem, the tendency is to use it in an essential way. The following is a useful substitute for the bounded convergence theorem stated in the language of linear space theory.

Theorem 1. Let $C$ be a compact (or countably compact) subset of a (real or complex) linear topological space $E$, and let $\{f_n\}$ be a sequence of continuous linear functionals on $E$ which is uniformly bounded on $C$. If, for each $x$ in $C$, $\lim_{n \to \infty} f_n(x) = 0$, then the same equality holds for every $x$ in the closed convex extension of $C$.

In case $C$ is compact and Hausdorff, the proof of Theorem 1 may run as follows: Let $F$ be the Banach space of all scalar-valued continuous functions on $C$ with the supremum norm; then there is a linear transformation $T$ on the dual $E^*$ of $E$ into $F$ defined by the equation $T(f) = f|_C$. Let $x_0$ be a point in the closed convex extension of $C$. Then one can define a bounded functional $\phi$ on the range of $T$.

Received by the editors October 20, 1960.