ON THE SPECTRUM OF A CONTRACTION

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1. Introduction. In this note we present several results on the spectrum of a contraction. The first is an extension to the approximate point spectrum of a result of Nagy and Foias, on the relation of the point spectrum of a contraction and that of its unitary dilation, which has several corollaries. The second is a simple solution to a problem in spectral mapping raised in [2]. Finally we have a result on the point spectrum of a class of contractions discussed in [3]. For the background on unitary dilations see [4] or [5].

2. Arbitrary contractions. In Theorem 1 of [6] it is shown that the set of eigenvalues of modulus 1 of a contraction $A$ coincides with that of its unitary dilation $U$. Less is true for the approximate point spectrum $\Sigma_{ap}A$. (See [1] for the definition of $\Sigma_{ap}$.)

Proposition. Let $A$ be a contraction on a Hilbert space $H$ and let $U$ be a unitary dilation on a (larger) space $K$. Then $\mu = e^{i\alpha} \in \Sigma_{ap} A$ if and only if $\mu \in \Sigma_{ap} U$ with approximate eigenvectors in $H$.³

(Thus, if $\mu \in \Sigma_{ap} U$, $|\mu| = 1$, but the approximate eigenvectors are not in $H$, then $\mu \not\in \Sigma_{ap} A$.)

Proof. Let $P$ be the projection of $K$ onto $H$. If there are unit vectors $x_n \in H$ with $\|Ux_n - \mu x_n\| \rightarrow 0$ as $n \rightarrow \infty$, then $\|Ax_n - \mu x_n\| = \|Pu_n - \mu x_n\| \leq \|Ux_n - \mu x_n\| \rightarrow 0$ as $n \rightarrow \infty$, so that $\mu \in \Sigma_{ap} A$. For the converse, there is clearly no loss of generality in taking $\mu = 1$, and we suppose there are unit vectors $x_n \in H$ such that

$$\|Ax_n - x_n\| \leq 1/n, \quad n = 1, 2, \ldots,$$

from which it follows that $\|Ax_n\| \geq 1 - 1/n$. Again let $P$ be the projection of $K$ onto $H$, and write $H^\perp$ for the orthogonal complement of $H$ in $K$. Now $Ux_n = u_n + v_n$, with $u_n \in H$, $v_n \in H^\perp$, and $\|u_n\|^2 + \|v_n\|^2 = \|u_n\|^2 = \|x_n\|^2 = 1$. Since $u_n = Pu_n = Ax_n$, we have

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³ By a unitary dilation of an operator $A$ on $H$ is meant a unitary operator $U$ on a space $K \supseteq H$ such that $PUx = Ax$ for all $x \in H$, where $P$ is the projection of $K$ onto $H$. In [4, 5] a unique minimal such dilation is studied, but for present purposes minimality is irrelevant.
1 = \| A x_n \|^2 + \| v_n \|^2 \geq 1 - \frac{2}{n} + \frac{1}{n^2} + \| v_n \|^2,

so that

\[ \| v_n \|^2 \leq \frac{2}{n} - \frac{1}{n^2}. \]

The components in \( H \) and \( H_1 \) of \( U x_n - x_n \), the vector whose norm is to be estimated, are \( A x_n - x_n \) and \( v_n \), respectively, as is clear, and so

\[ \| U x_n - x_n \|^2 = \| A x_n - x_n \|^2 + \| v_n \|^2 \leq \frac{1}{n^2} + \frac{2}{n} - \frac{1}{n^2} = \frac{2}{n}, \]

by the displayed inequalities, whence \( \| U x_n - x_n \| \leq (2/n)^{1/2} \) and the proof is complete.

[Added in proof. Professor Sz.-Nagy has remarked in a private communication that the proposition may be proved very simply as follows. With notation as above, for \( x \in H \) we have

\[ \| U x - x \|^2 = \| U x \|^2 + \| x \|^2 - 2 \Re (U x, x) = 2 \| x \|^2 - 2 \Re (T x, x) = 2 \Re (x, x - T x) \leq 2 \| x \| \| x - T x \|, \]

and the conclusion follows at once.]

**Corollary 1.** The approximate eigenvalues of modulus 1 of \( A^* \) are the complex conjugates of those of \( A \).

**Proof.** If \( \mu = e^{i \alpha} \in \Sigma_{ap} A \) then \( \mu \in \Sigma_{ap} U \) with approximate eigenvectors in \( H \), so that given \( \epsilon > 0 \) there exists a unit vector \( x \in H \) with \( \| U x - \mu x \| < \epsilon \). Hence \( \| U^* x - \bar{\mu} x \| < \epsilon \), trivially, and by the proposition again it follows that \( \| A^* x - \bar{\mu} x \| < \epsilon \), as was to be shown.

The same result for the point spectrum is given in [5, p. 88].

**Corollary 2.** Near a gap in \( \Sigma U \) there can be only residual spectrum of \( A \).

**Proof.** By a gap in \( \Sigma U \) is meant an open arc of the unit circle which lies in the complement of \( \Sigma U \), and the assertion is that every such gap is contained in a planar open set disjoint from \( \Sigma_{ap} A \). The proof is based on the closure of \( \Sigma_{ap} A \). Suppose this for the moment. Let \( G \) be a gap in \( \Sigma U \) and \( e^{i \alpha} \in G \). Then there must be an open circle \( C_\alpha \) centered at \( e^{i \alpha} \) with \( C_\alpha \cap \Sigma_{ap} A = \emptyset \), else \( e^{i \alpha} \) would be a limit point of \( \Sigma_{ap} A \), hence in \( \Sigma_{ap} A \), and therefore by the proposition a member of \( \Sigma_{ap} U \), contrary to supposition. The open set required by the corollary is then \( U_\alpha C_\alpha \). We complete the proof by showing that \( \Sigma_{ap} A \) is closed,
for any bounded $A$. Let $\lambda_n \in \Sigma_{ap}A$, $\lambda_n \to \lambda$. If $\lambda \notin \Sigma_{ap}A$ then there exists $\varepsilon > 0$ such that $\| (A - \lambda I)x \| \geq \varepsilon$ for all unit vectors $x$. Then $|\lambda - \lambda_n| = \| (A - \lambda I)x - (A - \lambda_n I)x \| \geq \varepsilon - \| (A - \lambda_n I)x \| \geq \varepsilon - |\lambda - \lambda_n|$ for all unit vectors $x$. In particular, if $|\lambda_n - \lambda| \leq \varepsilon/2$ then $\| (A - \lambda_n I)x \| \geq \varepsilon/2$ for all unit vectors, so that $\lambda_n \in \Sigma_{ap}A$, contrary to supposition.

A side condition such as the one employed in the proposition (that the approximate eigenvectors for $U$ be in $H$) is seen to be necessary by taking for $A$ any contraction with no spectrum on $\{ \| z \| = 1 \}$, whereas $\Sigma U \subseteq \{ \| z \| = 1 \}$ and $\Sigma U = \Sigma_{ap}U$ (see [1, p. 51]). In particular we know that for such $A$ the approximate eigenvectors of its dilation $U$ cannot be in $H$.

3. A spectral mapping problem. In [2] we studied the preservation of $\Sigma_{ap}A$ under general mappings and noted that in general it is not preserved in the reverse direction (that is, $\alpha \in f^{-1}(\beta)$ need not be an eigenvalue of $A$ when $\beta$ is an eigenvalue of $f(A)$). On the other hand it is trivially clear that if for all function $f$ the number $f(\mu)$ is an eigenvalue of $f(A)$ then $\mu$ is an eigenvalue of $A$. The problem is to find a nonvacuous condition sufficient for preservation of $\Sigma_{ap}A$ in the reverse direction.

Let $C_n(f)$ be the $n$th Taylor coefficient of $f$, and write $f_t$ for the function $f_t(s) = f(ts)$.

**Proposition.** Let $A$ be a contraction, and $f$ a fixed function analytic for $\| z \| < 1$. If $f_t(A)x = f_t(\mu)x$ for infinitely many (complex) $t$ converging inside the unit circle, then $A^m x = \mu^m x$, where $m$ is the least $n > 0$ such that $C_n(f) \neq 0$. Conversely, if $A^m x = \mu^m x$ and $C_k(f) = 0$ for $0 \leq k < m$ then $f_t(A)x = f_t(\mu)x$ for all $|t| < 1$.

**Proof.** By hypothesis $f(z) = \sum_0^\infty C_n(f) z^n$ converges for $\| z \| < 1$, so $f_t(z) = \sum_0^\infty C_n(f) t^n z^n$ has radius of convergence $r(t) > 1$ for $|t| < 1$. Since $\| A \| \leq 1$ the operator series $\sum_0^\infty C_n(f) A^n t^n$ converges in norm, for $|t| < 1$, to an operator which we define as $f_t(A)$, so that

$$(f_t(A)x, y) = \sum_0^\infty C_n(f)(A^nx, y)t^n = F(t), \quad |t| < 1,$$

is an analytic function of $t$, for each pair $x, y$ of vectors. (This definition of $f_t(A)$ agrees with that of [4], $(f_t(A)x, y) = \int f_t(e^{it})dF(s)x, y$), for by uniform convergence the integral is equal to

$$\sum_0^\infty C_n(f)t^n \int e^{ins}dF(s)x, y = \sum C_n(f)t^n(A^nx, y),$$
and it is easy to see also that it agrees with the classical definition by the Cauchy integral formula. Similarly \( f_s(\mu)(x, y) = (f_s(\mu)x, y) \) may be expanded in the series

\[
(f_s(\mu)x, y) = \sum_{n=0}^{\infty} C_n(f)(\mu^n x, y) t^n = G(t), \quad |t| < 1.
\]

Now for the first assertion of the proposition we have by hypothesis that \( F = G \) on an infinite set with limit point inside the circle. Since \( F \) and \( G \) are clearly analytic for \( |t| < 1 \) we conclude that \( F(t) = G(t) \), \( |t| < 1 \). This means that, for all \( n \geq 0 \) and all \( y \in H \),

\[
c_n(f)(A^nx, y) = c_n(f)(\mu^n x, y),
\]

and the assertion now follows by cancellation of \( c_n(f) \). The second assertion goes in the same spirit. The hypotheses involve \( F(t) = G(t) \) for \( |t| < 1 \) and therefore \( f_s(A)x, y) = (f_s(\mu)x, y) \) for all \( y \) and \( |t| < 1 \), which yields the conclusion.

4. A contraction \( A \) is absolutely continuous if there exists a function \( K(t, x, y) \in L_1(0, 2\pi) \) for every pair of vectors \( x, y \), such that

\[
(A^{(n)}x, y) = \frac{1}{2\pi} \int_0^{2\pi} e^{int} K(t, x, y) dt
\]

for all \( n = 0, \pm 1, \pm 2, \cdots \) (here \( A^{(-n)} = A^{*n} \) (see [3])). This is a smoothness condition which reflects itself in the spectrum of \( A \) as follows:

**Proposition.** An absolutely continuous contraction has no eigenvalues of modulus 1.

**Proof.** Let \( A \) be absolutely continuous. The representation above for \( A \) in terms of \( K \) amounts to the assertion that \( K \) has the Fourier expansion

\[
K(t, x, y) \sim \sum_{n=-\infty}^{\infty} e^{-int}(A^{(n)}x, y).
\]

Now suppose that \( Ax = e^{i\beta}x \) for some unit vector \( x \) and \( 0 \leq \beta \leq 2\pi \). It then follows from [5, p. 88] that \( A^{(n)}x = e^{in\beta}x \) for \( n = 0, \pm 1, \pm 2, \cdots \). Hence the Fourier expansion for \( K(\cdot, x, x) \) reduces to

\[
K(t, x, x) \sim \sum_{n=-\infty}^{\infty} e^{in\beta} e^{-int}.
\]

But \( K(\cdot, x, x) \in L_1 \), so that its Fourier coefficients \( e^{in\beta} \) must tend to 0. This contradiction completes the proof.
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1. Lebesgue's bounded convergence theorem has become a powerful tool in the theory of linear topological spaces, and recently, for a treatment of weak convergence of sequences or for a proof of Krein's theorem, the tendency is to use it in an essential way.¹ The following is a useful substitute for the bounded convergence theorem stated in the language of linear space theory.

**Theorem 1.** Let $C$ be a compact (or countably compact)² subset of a (real or complex) linear topological space $E$, and let $\{f_n\}$ be a sequence of continuous linear functionals on $E$ which is uniformly bounded on $C$. If, for each $x$ in $C$, $\lim n f_n(x) = 0$, then the same equality holds for every $x$ in the closed convex extension of $C$.

In case $C$ is compact and Hausdorff, the proof of Theorem 1 may run as follows: Let $F$ be the Banach space of all scalar-valued continuous functions on $C$ with the supremum norm; then there is a linear transformation $T$ on the dual $E^*$ of $E$ into $F$ defined by the equation $T(f) = f|_C$. Let $x_0$ be a point in the closed convex extension of $C$. Then one can define a bounded functional $\phi$ on the range of $T$

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¹ I am indebted to the referee for the remark that, in Dunford and Schwartz [2], Krein's theorem is proved using Riesz-Markoff-Kakutani's theorem but not Lebesgue's bounded convergence theorem. Their proof relies on the theory of integration of vector-valued functions.

² A space $X$ is countably compact if each sequence in $X$ has a cluster point.