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THE GEÖCZE $k$-AREA AND A CYLINDRICAL PROPERTY

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In [5], a definition of the Geöcze $k$-area of a mapping from admissible sets of Euclidean $k$-space $E_k$ into Euclidean $n$-space $E_n$ ($2 \leq k \leq n$) is given. This definition is an extension of the Geöcze area given in [3]. With this definition of Geöcze $k$-area, a treatment of Geöcze $k$-area is developed for flat mappings ($k = n$) paralleling the treatment of Geöcze area for plane mappings given in [3]. The present paper gives results concerning the Geöcze $k$-area for mappings from admissible sets of $E_k$ into $E_n$ ($n > k \geq 2$). A cylindrical property is defined for mappings in harmony with [3, (16.10)]. This property, which has had an essential part in the proofs of the main theorems for Lebesgue area for mappings from admissible sets of $E_2$ into $E_n$ [3] and which has been used in other research, is shown to play a prominent role in the extension of the theory of Geöcze area to higher dimensions. An example is given to show that the theorems concerning the cylindrical property in [3] are no longer valid for $k \geq 3$. These theorems are shown to be valid under a certain restrictive hypothesis found in the literature.

1. Notations and definitions. If $X$ is a set in $E_k$, then $\overline{X}$, $X^0$, and $X^*$ will denote respectively the closure, interior, and boundary of $X$.

A polyhedral region $R$ in $E_k$ is the point-set covered by a strongly connected $k$-complex situated in $E_k$. A polyhedral region $R$ is called simple if $E_k - R$ is connected (see [5]).

By a figure $F$ we mean a finite union of nonoverlapping polyhedral regions in $E_k$ such that the interior of the union is the union of the interior of the finitely many polyhedral regions. A set $A$ in $E_k$ is said to be admissible in each of the following cases:

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(a) $A$ is an open set in $E_k$.
(b) $A$ is a figure.
(c) $A$ is a homeomorphic image of a figure.
(d) $A$ is a set open in the type (b) or (c) above.

By $3(k, n)$ we mean the set of all continuous mappings $(T, A)$ from admissible sets $A$ in $E_k$ into $E_n$ $(2 \leq k \leq n)$. A continuous mapping $(T, A)$ is called flat if $(T, A) \in 3(k, k)$.

Let $(T_0, A_0), (T_j, A_j) (j = 1, 2, \cdots)$ be a sequence of mappings in $3(k, n)$. $(T_j, A_j) (j = 1, 2, \cdots)$ is said to converge to $(T_0, A_0)$ if

(i) $A_j \subset A_{j+1} \subset A_0$ for all $j$;
(ii) $\bigcup_{j=1}^{\infty} A_j = A_0$;
(iii) $\sup \{ |T(w) - T_j(w)| : w \in A_j \} \to 0$ as $j \to \infty$.

If $k < n$ are positive integers then $\mathcal{O}_n^k$ is the set of all $k$-termed sequences $\xi = (\xi_1, \xi_2, \cdots, \xi_k)$ of integers such that $1 \leq \xi_1 < \xi_2 < \cdots < \xi_k \leq n$. We shall assume that $\mathcal{O}_n^k$ is lexicographically ordered.

If $x = (x^1, x^2, \cdots, x^n) \in E_n$ then by the mapping $P_n^x$, $x \in \mathcal{O}_n^k$, we shall mean the projection of $E_n$ onto the coordinate hyperspace $E_n^x$ where $P_n^x(x) = (x^1, x^2, \cdots, x^n) \in E_n^x$.

If $(T, A) \in 3(k, n)$ then by $(T^k, A^k)$ we mean the mapping $(P_n^x T, A), \xi \in \mathcal{O}_n^k, k \leq m \leq n$. If $m = k$ then $(T^k, A), \xi \in \mathcal{O}_n^k$, forms a collection of $C^k$ flat mappings from the admissible set $A$ into $E_n^x \subset E_n, \xi \in \mathcal{O}_n^k$.

Let $(T, A) \in 3(k, n)$ and $D$ be any bounded open connected set such that $D \subset A$. If $\mathcal{O}_n^k(T^k, A^k), \xi \in \mathcal{O}_n^k$, is a compact set, then a topological index $O(x; C^k), x \in E_n^x$, is defined {see [5, (1.3)] and also [6, p. 125]}. The function $O(x; C^k)$ is a measurable function of $x \in E_n^x$ and the integral $\nu(T^k, D) = \{E_n^x \int O(x; C^k) \}$ exists (finite or infinite) for each $\xi \in \mathcal{O}_n^k$. $[\nu(T^k, D) : \xi \in \mathcal{O}_n^k]$ forms a $1 \times \mathbb{R}$ vector. Denote by $v(T, D)$ the Euclidean norm of this vector; i.e., $v(T, D) = \{ \sum \nu^2(T^k, D) \}^{1/2}$, where $\sum$ ranges over $\xi \in \mathcal{O}_n^k$. Denote by $\mathcal{S}$ a finite system of nonoverlapping simple polyhedral regions $\pi \subset A$. We define the geometric k-area of $(T, A)$ to be the number $V(T, A) = \sup \{ v(T, \pi) : \pi \subset A \}$, where $v(T, \pi) = v(T, \pi^0), \sum$ ranges over $\pi \in \mathcal{S}$ and the supremum is taken over all possible systems $\mathcal{S}$. If $V(T, A) < \infty$, then the integral $u(T^k, D) = \{E_n^x \int O(x; C^k) \}$ exists for each $\xi \in \mathcal{O}_n^k$. Let $u(T, D) = \{ \sum u^2(T^k, D) \}^{1/2}$, and $U(T, A) = \sup \{ u(T, \pi) : \pi \subset A \}$, where $u(T, \pi) = u(T, \pi^0)$ and the sum $\sum$ and the supremum are taken as above.

Let $(T, A) \in 3(k, n)$. Then $\Gamma(T, A)$ will denote the collection of all maximal connected sets of $A$ on which $(T, A)$ is constant. If $A$ is compact then $\Gamma(T, A)$ forms an upper semicontinuous decomposition of $A$.

2. The cylindrical property. In this section we confine ourselves to continuous mappings $(T, A) \in 3(k, n)$ with $A$ compact and $n > k$.
such mappings we have $C^*_d$ flat mappings $(T^*, A)$, $\xi \in \Omega^n_d$. Associated with these mappings are their corresponding upper semicontinuous decompositions $\Gamma(T, A)$, $\Gamma(T^*, A)$, $\xi \in \Omega^n_d$, of $A$. A point $x \in E^*_d$ is said to have the cylindrical property with respect to $(T, A)$ if (i) $x \in T^*(A)$; (ii) there is a $g \in \Gamma(T^*, A)$ such that $T^*(g) = x$ and $T(g)$ is nondegenerate.

2.1. Theorems concerning the cylindrical property. For each $\xi \in \Omega^n_d$ let $X^\xi$ be the set of all $x \in E^*_d$ having the cylindrical property with respect to $(T, A)$.

**Theorem (i).** Let $(T, A) \in \mathcal{G}(2, n)$ with $A$ compact and $V(T, A) < \infty$. Then, for each $\xi \in \Omega^n_d$, $X^\xi$ has zero Lebesgue 2-measure.

**Proof.** This theorem was first proved by L. Cesari in [2] for the case $n = 3$. It is easily shown that this special case implies the theorem. Another proof is also given in [4, Theorem 8.12].

Theorem (i) above shows that for $k = 2$, the condition $V(T, A)$ is finite has very strong implications, namely, the set $X^\xi$ has zero measure for all $\xi \in \Omega^n_d$. This no longer is true when $k > 2$ as an example in (2.2) below shows. The following theorem gives a sufficient condition on the mapping $(T, A) \in \mathcal{G}(k, n)$ to make the Lebesgue $k$-measure of $X^\xi$ zero, $\xi \in \Omega^n_d$, $n > k \geq 2$.

**Theorem (ii).** Let $(T, A) \in \mathcal{G}(k, n)$ ($2 \leq k < n$), $A$ be compact and $H^{k+1}[T(A)] = 0$, where $H^{k+1}[\cdot]$ denotes the $(k+1)$-dimensional Hausdorff measure in $E_n$. Then $m_k[X^\xi] = 0$ for all $\xi \in \Omega^n_d$, where $m_k[\cdot]$ is the $k$-dimensional Lebesgue measure.

**Proof.** Let $\alpha \in \Omega^{k+1}_n$, $P^*_n$ and $E^*_{k+1}$ be defined as in §1 above. Let us fix a $\xi \in \Omega^n_d$. For each $\alpha \in \Omega^{k+1}_n$ with $E^*_d \subset E^*_{k+1}$, let $X^\xi_\alpha$ denote the set of all points of $E^*_d$ which have the cylindrical property with respect to the mapping $(P^*_n T, A)$. Then we have that $X^\xi = \bigcup X^\xi_\alpha$, where $\mathcal{U}$ ranges over all $\alpha \in \Omega^{k+1}_n$ with $E^*_d \subset E^*_{k+1}$. Hence it is sufficient to show that $m_k[X^\xi_\alpha] = 0$ for each $\alpha \in \Omega^{k+1}_n$ with $E^*_d \subset E^*_{k+1}$.

Since $0 = H^{k+1}[T(A)] \geq H^{k+1}[P^*_n T(A)] = m_{k+1}[P^*_n T(A)] \geq 0$, $\alpha \in \Omega^{k+1}_n$, we have $m_{k+1}[P^*_n T(A)] = 0$ for each $\alpha \in \Omega^{k+1}_n$. This implies that we need only study the case where $(T, A)$ is a continuous mapping of the compact admissible set $A \subset E_k$ into $E_{k+1}$ with $m_{k+1}[T(A)] = 0$. Suppose this is the case and let $\xi \in \Omega^{k+1}_n$. For the sake of simplicity, let $\xi$ be such that $E^*_d = \{ x \in E_{k+1} : x^i = 0 \}$. Let $[a_i, b_i]$, $i = 1, 2, \ldots$, be the countable collection of closed intervals on the $x^i$-axis of $E_{k+1}$, where $a_i$ and $b_i$ are rational numbers, $a_i < b_i$. Denote by $P$ the usual projection of $E_{k+1}$ onto the $x^i$-axis. If $X^\xi_\alpha$ denotes the set of all $x \in X^\xi$ with the property that $(T^*)^{-1}(x)$ has a component $g$ such that $PT(g)$
covers \([a_i, b_i]\), then \(X^i = \bigcup_{i=1}^{n_i} X_i^i\). Since \(A\) is compact and \(\Gamma(T, A)\) is an upper semicontinuous decomposition of \(A\), \(X_i^i\) is closed and hence compact for each \(i\).

Let \(Z_i = T(A) \cap (P_{k+1})^{-1}(X_i^i)\). \(Z_i\) is compact and hence measurable. Hence for each \(i\), \(0 = m_{k+1}[T(A)] \geq m_{k+1}[Z_i] \geq m_k[X_i^i] \cdot (b_i - a_i) \geq 0\). Therefore \(m_k[X_i^i] = 0\). Thus we conclude that \(m_k[X_i^i] = 0\) and the theorem is proved.

2.2. Example. We give an example to show that in (2.1.ii) above, the condition \(H_k^{k+1}[T(A)] = 0\) cannot be replaced by \(V(T, A) < \infty\) if \(n > k = 3\). This example will be used again in the next section.

Let \(A = \{w \in E_3: 0 \leq w_1 \leq 1, \ i = 1, 2, 3, A_0 = \{w \in A: w_3 = 0\}\), and let \(P_0\) be the usual projection map of \(A\) onto \(A_0\). To facilitate discussion, we will denote the \(m\)-dimensional hyperspace of \(E_3\) spanned by \((x_1, x_2, \ldots, x_m, 0, \ldots, 0)\) by \(E(x_1, x_2, \ldots, x_m)\).

Let \((T_0, A_0): x^i = f_i(w_1, w_2), w \in A, i = 1, 2, 3, x^4 = w_3, w = (w_1, w_2, w_3) \in A\). Clearly \((T, A)\) is a homeomorphism. Hence every element of \(T(T, A)\) is a point \(w \in A\). Let \(\xi \in \Omega^e_m\) be such that \(E_3^m = E(x_1, x_2, x^4)\). Then every element \(g \in \Gamma(T, A)\) is an interval. Hence \(X^i = T^i(A), m_k[X^i] = m_k[T^i(A)], m_k[T^i(A)] > 0\), since \(T^i(w) = T^i P_0(w)\).

Let us show \(V(T, A) \leq \varepsilon\). Since the 2-area of \((T_0, A_0)\) is \(\varepsilon\), by \([3, (24.1.i)]\), there exists a sequence of quasi-linear mappings \((Q_{ij}, A_0): x^i = h^i_j(w_1, w_2), i = 1, 2, 3, w \in A, j = 1, 2, \ldots,\) converging to \((T_0, A_0)\) and the 2-area of \((Q_{ij}, A_0)\) converging to \(\varepsilon\). Let \((Q_j, A)\) be the quasi-linear mapping \(x^i = h^i_j(w_1, w_2), i = 1, 2, 3, x^4 = w_3, w \in A, j = 1, 2, \ldots\). Clearly, \((Q_j, A) \rightarrow (T, A)\) as \(j \rightarrow \infty\). By \([5, (2.2.x)]\), \(V(Q_j, A)\) is equal to the 2-area of \((Q_{ij}, A_0)\). Hence by the lower semicontinuity of the Geöcze \(k\)-area, \(\varepsilon = \lim_{j \rightarrow \infty} V(Q_j, A) \geq V(T, A)\) \([5, (2.2.vi)]\). Thus the example is constructed.

3. The indices \(d, m, \sigma\). In the theory of Geöcze area, the indices \(d, m, \sigma\) play an important role (see \([3]\)). These indices have been defined for the Geöcze \(k\)-area \([5, (3.3)]\). We give these definitions below.

Let \((T, F) \in \Omega(k, n)\) where \(F\) is a figure. Denote by \(\Xi = \Xi' \cup \Xi''\) a finite subdivision of \(F\) into nonoverlapping polyhedral regions, where \(\Xi'\) is the collection of simple polyhedral regions and \(\Xi''\) is the collection of nonsimple polyhedral regions. Then \(d = \max \{\dim(T(R)): R \in \Xi\}\), \(m = \max \{m_k[U T^i(R^\ast)]: \xi \in \Omega^e_m\}\), where \(U\) ranges over all \(R \in \Xi'\) with \(R \subset F^3\). \(\sigma = \max \{\sigma^\xi: \xi \in \Omega^e_m\}, \sigma^\xi = \sum \sum \ast \ast \dim(T^i(B_i)), \agenda\sum \sigma^\xi = \sum \sum \ast \ast \dim(T^i(B_i))\), with \(\sum \ast \ast \
THEOREM (i). If \( k = 2 \) and \( V(T, F) < \infty \), then for every \( \epsilon > 0 \) there is a subdivision \( \mathcal{G} = \mathcal{G}' \cup \mathcal{G}'' \) of \( F \) with indices \( d, m, \sigma < \epsilon \).

THEOREM (ii). If \( k \geq 2 \) and \( H_n^{k+1}[T(F)] = 0 \), then for every \( \epsilon > 0 \) there is a subdivision \( \mathcal{G} = \mathcal{G}' \cup \mathcal{G}'' \) of \( F \) with indices \( d, m, \sigma < \epsilon \).

Proof. The proofs of Theorems (i) and (ii) are essentially the same as the proof of [3, (21.1.i)]. The only change needed is the replacement of [3, (16.10.i)] by (2.1.i) and (2.1.ii), respectively.

We observe that in the example of (2.2), \( d + m + \sigma > \epsilon_0 > 0 \) for an appropriate \( \epsilon_0 \), and \( H_n^{k+1}[T(F)] \neq 0 \).

4. Theorems on limits with respect to \( d, m, \sigma \). Let \( (T, A) \in \mathcal{I}(k, n) \), \( F_j \ (j = 1, 2, \ldots) \) be a sequence of figures with \( F_j \subset F_{j+1} \subset A \) and \( \bigcup_{j=1}^{\infty} F_j = A_0 \), and \( \mathcal{G}_j = \mathcal{G}_j' \cup \mathcal{G}_j'' \) be a subdivision of \( F_j \) with indices \( d_j, m_j, \sigma_j \ (j = 1, 2, \ldots) \).

THEOREM (i). If \( A_0 \subset A \) is an admissible set and \( d_j + m_j + \sigma_j \to 0 \) as \( j \to \infty \), then

\[
\lim_{j \to \infty} \sum_{i} v(T, q) = V(T, A_0), \quad \lim_{j \to \infty} \sum_{i} v(T^i, q) = V(T^i, A_0), \quad \xi \in \Omega^k,
\]

where \( \sum_{i} \) ranges over all \( q \in \mathcal{G}_j, q \subset A_0 \).

If, furthermore, \( V(T, A) < \infty \), then

\[
\lim_{j \to \infty} \sum_{i} u(T, q) = U(T, A_0) = V(T, A_0),
\]

\[
\lim_{j \to \infty} \sum_{i} |u(T^i, q)| = U(T^i, A_0) = V(T^i, A_0), \quad \xi \in \Omega^k,
\]

where \( \sum_{i} \) ranges over all \( q \in \mathcal{G}_j, q \subset A_0 \).

Proof. The proof of (i) is essentially the same as [3, (21.2.i), (21.3.i, ii)].

From (3.i, ii) we have sufficient conditions on \( (T, A) \) for the existence of subdivisions \( \mathcal{G} = \mathcal{G}' \cup \mathcal{G}'' \) of \( F \subset A \) with arbitrary small indices \( d, m, \sigma \). Namely, (1) \( k = 2 \) and \( V(T, A) < \infty \); or (2) \( k \geq 2 \) and \( H_n^{k+1}[T(A)] = 0 \).

5. In this section we give some theorems concerning the Geöcze \( k \)-area of mappings \( (T, A) \in \mathcal{I}(k, n) \).

THEOREM (i). If \( (T, A) \in \mathcal{I}(k, n) \) and \( H_n^{k+1}[T(A)] = 0 \), then
\[ \left( \sum V^2(T^\xi, A) \right)^{1/2} \leq V(T, A) \leq \sum V(T^\xi, A), \text{ where } \sum \text{ ranges over } \xi \in \Omega_n. \]

**Theorem (ii).** Let \((T, A) \in \mathcal{A}(k, n)\) with \(H^{n+1}_n[T(A)] = 0\) and \(H \subset A\) be a set closed in \(A\) such that \(m_n[T^\xi(A^\circ \cap H)] = 0, \xi \in \Omega_n^k.\) Then each component \(A_i\) of \(A - H\) is an admissible set and \(V(T, A) = \sum V(T, A_i), V(T^\xi, A) = \sum V(T^\xi, A_i), \xi \in \Omega_n^k,\) where \(\sum \text{ ranges over all components } A_i \text{ of } A - H.\)

**Theorem (iii).** Let \((T, A) \in \mathcal{A}(k, n)\) and \(H^{n+1}_n[T(A)] = 0.\) Let \(A_i \subset A (i = 1, 2, \ldots)\) be a collection of sets open in \(A\) such that each \(A_i\) is the union of sets \(g \in \Gamma(T, A).\) Then we have \(V(T, \bigcup A_i) \leq \sum V(T, A_i)\) and \(V(T^\xi, \bigcup A_i) \leq \sum V(T^\xi, A_i), \xi \in \Omega_n^k, \text{ where } \bigcup \text{ and } \sum \text{ range over all } A_i.\)

The proofs of the above theorems are essentially the same as \([3, (21.4.i), (21.5.i), (25.1.iii)].\)

**Bibliography**