THE GEÖCZE $k$-AREA AND A CYLINDRICAL PROPERTY

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In [5], a definition of the Geöcze $k$-area of a mapping from admissible sets of Euclidean $k$-space $E_k$ into Euclidean $n$-space $E_n$ ($2 \leq k \leq n$) is given. This definition is an extension of the Geöcze area given in [3]. With this definition of Geöcze $k$-area, a treatment of Geöcze $k$-area is developed for flat mappings ($k = n$) paralleling the treatment of Geöcze area for plane mappings given in [3]. The present paper gives results concerning the Geöcze $k$-area for mappings from admissible sets of $E_k$ into $E_n$ ($n > k \geq 2$). A cylindrical property is defined for mappings in harmony with [3, (16.10)]. This property, which has had an essential part in the proofs of the main theorems for Lebesgue area for mappings from admissible sets of $E_k$ into $E_n$ [3] and which has been used in other research, is shown to play a prominent role in the extension of the theory of Geöcze area to higher dimensions. An example is given to show that the theorems concerning the cylindrical property in [3] are no longer valid for $k \geq 3$. These theorems are shown to be valid under a certain restrictive hypothesis found in the literature.

1. Notations and definitions. If $X$ is a set in $E_k$, then $\overline{X}$, $X^0$, and $X^*$ will denote respectively the closure, interior, and boundary of $X$.

A polyhedral region $R$ in $E_k$ is the point-set covered by a strongly connected $k$-complex situated in $E_k$. A polyhedral region $R$ is called simple if $E_k - R$ is connected (see [5]).

By a figure $F$ we mean a finite union of nonoverlapping polyhedral regions in $E_k$ such that the interior of the union is the union of the interior of the finitely many polyhedral regions. A set $A$ in $E_k$ is said to be admissible in each of the following cases:

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(a) $A$ is an open set in $E_k$.
(b) $A$ is a figure.
(c) $A$ is a homeomorphic image of a figure.
(d) $A$ is a set open in the type (b) or (c) above.

By $3(k, n)$ we mean the set of all continuous mappings $(T, A)$ from admissible sets $A$ in $E_k$ into $E_n$ ($2 \leq k \leq n$). A continuous mapping $(T, A)$ is called flat if $(T, A) \in 3(k, k)$.

Let $(T_0, A_0), (T_j, A_j)$ $(j=1, 2, \ldots)$ be a sequence of mappings in $3(k, n)$. $(T_j, A_j)$ $(j=1, 2, \ldots)$ is said to converge to $(T_0, A_0)$ if

(i) $A_j \subset A_{j+1} \subset A_0$ for all $j$;
(ii) $U_{j-1}^{n} A_j = A_0$;
(iii) $\sup \{ |T(w) - T_j(w)| : w \in A_j \} \to 0$ as $j \to \infty$.

If $k < n$ are positive integers then $\Omega^k_n$ is the set of all $k$-termed sequences $\xi = (\xi_1, \xi_2, \ldots, \xi_k)$ of integers such that $1 \leq \xi_1 < \xi_2 < \cdots < \xi_k \leq n$. We shall assume that $\Omega^k_n$ is lexicographically ordered.

If $x = (x_1, x_2, \ldots, x^n) \in E_n$ then by the mapping $P^k_x$, $x \in \Omega^k_n$, we shall mean the projection of $E_n$ onto the coordinate hyperspace $E^k_x$ where $P^k_x(x) = (x_1, x_2, \ldots, x_k) \in E^k_x$.

If $(T, A) \in 3(k, n)$ then by $(T^k, A)$ we mean the mapping $(P^k_x T, A)$, $x \in \Omega^k_n$, $k \leq m \leq n$. If $m = k$ then $(T^k, A)$, $x \in \Omega^k_n$, forms a collection of $C^k_n$ flat mappings from the admissible set $A$ into $E^k_x \subset E_n$, $x \in \Omega^k_n$.

Let $(T, A) \in 3(k, n)$ and $D$ be any bounded open connected set such that $\overline{D} \subset A$. If $C^k : (T^k, A^k)$, $x \in \Omega^k_n$, then a topological index $O(x; C^k), x \in \Omega^k_n$, is defined (see [5, (1.3)] and also [6; p. 125]). The function $O(x; C^k)$ is a measurable function of $x \in E^k_x$ and the integral $\nu(T^k, D) = (E^k_x) \int O(x; C^k)$ exists (finite or infinite) for each $x \in \Omega^k_n$.

$[\nu(T^k, D) : x \in \Omega^k_n]$ forms a $1 \times C^k_n$ vector. Denote by $\nu(T, D)$ the Euclidean norm of this vector; i.e., $\nu(T, D) = \left[ \sum \nu^2(T^k, D) \right]^{1/2}$, where $\sum$ ranges over $x \in \Omega^k_n$. Denote by $\mathcal{S}$ a finite system of nonoverlapping simple polyhedral regions $\pi \subset A$. We define the Geöse k-area of $(T, A)$ to be the number $V(T, A) = \sup \sum \nu(T, \pi)$, where $\nu(T, \pi)$ = $\nu(T, \pi^0)$, $\sum$ ranges over $\pi \in \mathcal{S}$ and the supremum is taken over all possible systems $\mathcal{S}$. If $V(T, A) < \infty$, then the integral $\nu(T^k, D)$ $= (E^k_x) \int O(x; C^k)$ exists for each $x \in \Omega^k_n$. Let $u(T, D) = \left[ \sum u^2(T^k, D) \right]^{1/2}$, and $U(T, A) = \sup \sum u(T, \pi)$, where $u(T, \pi) = u(T, \pi^0)$ and the sum $\sum$ and the supremum are taken as above.

Let $(T, A) \in 3(k, n)$. Then $\Gamma(T, A)$ will denote the collection of all maximal connected sets of $A$ on which $(T, A)$ is constant. If $A$ is compact then $\Gamma(T, A)$ forms an upper semicontinuous decomposition of $A$.

2. The cylindrical property. In this section we confine ourselves to continuous mappings $(T, A) \in 3(k, n)$ with $A$ compact and $n > k$. For
such mappings we have $O_n^k$ flat mappings $(T^i, A)$, $i \in \Omega^k_n$. Associated with these mappings are their corresponding upper semicontinuous decompositions $T(T^i, A)$, $\Gamma(T^i, A)$, $\xi \in \Omega^k_n$, of $A$. A point $x \in E^k_3$ is said to have the cylindrical property with respect to $(T, A)$ if (i) $x \in T^k(A)$; (ii) there is a $g \in \Gamma(T^i, A)$ such that $T^i(g) = x$ and $T(g)$ is nondegenerate.

2.1. Theorems concerning the cylindrical property. For each $\xi \in \Omega^k_n$ let $X^k_\xi$ be the set of all $x \in E^k_3$ having the cylindrical property with respect to $(T, A)$.

**Theorem (i).** Let $(T, A) \in 3(2, n)$ with $A$ compact and $V(T, A) < \infty$. Then, for each $\xi \in \Omega^k_n$, $X^k_\xi$ has zero Lebesgue 2-measure.

**Proof.** This theorem was first proved by L. Cesari in [2] for the case $n = 3$. It is easily shown that this special case implies the theorem. Another proof is also given in [4, Theorem 8.12].

Theorem (i) above shows that for $k = 2$, the condition $V(T, A)$ is finite has very strong implications, namely, the set $X^k_\xi$ has zero measure for all $\xi \in \Omega^k_n$. This no longer is true when $k > 2$ as an example in (2.2) below shows. The following theorem gives a sufficient condition on the mapping $(T, A) \in 3(k, n)$ to make the Lebesgue $k$-measure of $X^k_\xi$ zero, $\xi \in \Omega^k_n$, $n > k \geq 2$.

**Theorem (ii).** Let $(T, A) \in 3(k, n)$ $(2 \leq k < n)$, $A$ be compact and $H^{k+1}_n[T(A)] = 0$, where $H^{k+1}_n[\cdot]$ denotes the $(k + 1)$-dimensional Hausdorff measure in $E_n$. Then $m_k[X^k_\xi] = 0$ for all $\xi \in \Omega^k_n$, where $m_k[\cdot]$ is the $k$-dimensional Lebesgue measure.

**Proof.** Let $\alpha \in \Omega^k_{n+1}$, $P^\alpha_n$ and $E^\alpha_{k+1}$ be defined as in §1 above. Let us fix a $\xi \in \Omega^k_n$. For each $\alpha \in \Omega^k_{n+1}$ with $E^\alpha_k \subset E^\alpha_{k+1}$, let $X^\alpha_k$ denote the set of all points of $E^\alpha_k$ which have the cylindrical property with respect to the mapping $(P^\alpha_n T, A)$. Then we have that $X^k_\xi = \bigcup X^\alpha_k$, where $U$ ranges over all $\alpha \in \Omega^k_{n+1}$ with $E^\alpha_k \subset E^\alpha_{k+1}$. Hence it is sufficient to show that $m_k[X^\alpha_k] = 0$ for each $\alpha \in \Omega^k_{n+1}$ with $E^\alpha_k \subset E^\alpha_{k+1}$.

Since $0 = H^{k+1}_n[T(A)] \geq H^{k+1}_n[P^\alpha_n T(A)] = m_{k+1}[P^\alpha_n T(A)] \geq 0$, $\alpha \in \Omega^k_{n+1}$, we have $m_{k+1}[P^\alpha_n T(A)] = 0$ for each $\alpha \in \Omega^k_{n+1}$. This implies that we need only study the case where $(T, A)$ is a continuous mapping of the compact admissible set $A \subset E_k$ into $E_{k+1}$ with $m_{k+1}[T(A)] = 0$. Suppose this is the case and let $\xi \in \Omega^k_{n+1}$. For the sake of simplicity, let $\xi$ be such that $E^\xi_k = \{x \in E_{k+1}: x^1 = 0\}$. Let $[a_i, b_i]$, $i = 1, 2, \cdots$, be the countable collection of closed intervals on the $x^1$-axis of $E_{k+1}$, where $a_i$ and $b_i$ are rational numbers, $a_i < b_i$. Denote by $P$ the usual projection of $E_{k+1}$ onto the $x^1$-axis. If $X^\xi_k$ denotes the set of all $x \in X^k_\xi$ with the property that $(T^i)^{-1}(x)$ has a component $g$ such that $PT(g)$
covers \( [a_i, b_i] \), then \( X^\xi = \bigcup_{i=1}^{n^\xi} X_i^\xi \). Since \( A \) is compact and \( \Gamma(T^\xi, A) \) is an upper semicontinuous decomposition of \( A \), \( X_i^\xi \) is closed and hence compact for each \( i \).

Let \( Z_i = T(A) \cap (P_{k+1})^{-1}(X_i^\xi) \). \( Z_i \) is compact and hence measurable. Hence for each \( i \), \( 0 = \mu_{k+1}[T(A)] \geq \mu_{k+1}[Z_i] \geq \mu_k[X_i^\xi] \cdot (b_i - a_i) \geq 0 \). Therefore \( \mu_k[X_i^\xi] = 0 \). Thus we conclude that \( \mu_k[X^\xi] = 0 \) and the theorem is proved.

2.2. Example. We give an example to show that in (2.1.ii) above, the condition \( H_{k+1}^+ [T(A)] = 0 \) cannot be replaced by \( V(T, A) < \infty \) if \( n > k = 3 \). This example will be used again in the next section.

Let \( A = \{ w \in E^3 : 0 \leq w_i \leq 1, \quad i = 1, 2, 3 \}, \quad A_0 = \{ w \in A : w_3 = 0 \} \), and let \( P_0 \) be the usual projection map of \( A \) onto \( A_0 \). To facilitate discussion, we will denote the \( m \)-dimensional hyperspace of \( E_n \) spanned by \( (x_1, x_2, \ldots, x_m, 0, \ldots, 0) \) by \( E(x_1, x_2, \ldots, x_m) \).

Let \( (T_0, A_0) : x^i = f_i(w_1, w_2), \quad w \in A_0, \quad i = 1, 2, 3 \), be the continuous mapping defined by A. Besicovitch in [1] of the square \( A_0 \) into \( E(x_1, x_2, x_3) \) so that \( (T_0, A_0) \) is a homeomorphism, \( \mu_3[T_0(A_0)] > 0 \) and the 2-area of \( (T_0, A_0) \) is a positive number \( \epsilon \). Let \( (T, A) \) be defined by \( x^i = f_i(w_1, w_2), \quad i = 1, 2, 3 \), \( x^3 = w_3 \), \( w = (w_1, w_2, w_3) \in A \). Clearly \( (T, A) \) is a homeomorphism. Hence every element of \( \Gamma(T^\xi, A) \) is a point \( w \in A \). Let \( \xi \in \Omega_n^k \) be such that \( E^\xi = E(x_1, x_2, x_3) \). Then every element \( g \in \Gamma(T^\xi, A) \) is an interval. Hence \( X^\xi = T^\xi(A) \), and \( \mu_3[X^\xi] = \mu_3[T^\xi(A)] = \mu_3[T_0(A_0)] > 0 \), since \( T^\xi(w) = T_0P_0(w) \).

Let us show \( V(T, A) \leq \epsilon \). Since the 2-area of \( (T_0, A_0) \) is \( \epsilon \), by [3, (24.1.i)], there exists a sequence of quasi-linear mappings \( (Q_{0j}^\xi, A_0) : x^i = h_{0j}^i(w_1, w_2), \quad i = 1, 2, 3 \), \( w \in A_0, \quad j = 1, 2, \ldots \), converging to \( (T_0, A_0) \) and the 2-area of \( (Q_{0j}^\xi, A_0) \) converging to \( \epsilon \). Let \( (Q_{ij}, A) \) be the quasi-linear mapping \( x^i = h_{ij}^i(w_1, w_2), \quad i = 1, 2, 3 \), \( x^3 = w_3 \), \( w \in A \), \( j = 1, 2, \ldots \). Clearly, \( (Q_{ij}, A) \to (T, A) \) as \( j \to \infty \). By [5, (2.2.x)], \( V(Q_{ij}, A) \) is equal to the 2-area of \( (Q_{0j}^\xi, A_0) \). Hence by the lower semicontinuity of the Geöcze \( k \)-area, \( \epsilon = \lim_{j \to \infty} V(Q_{ij}, A) \geq V(T, A) \geq \delta \). Thus the example is constructed.

3. The indices \( d, m, \sigma \). In the theory of Geöcze area, the indices \( d, m, \sigma \) play an important role (see [3]). These indices have been defined for the Geöcze \( k \)-area [5, (3.3)]. We give these definitions below.

Let \( (T, F) \in \Omega(k, n) \) where \( F \) is a figure. Denote by \( \Omega = \Omega' \cup \Omega'' \) a finite subdivision of \( F \) into nonoverlapping polyhedral regions, where \( \Omega' \) is the collection of simple polyhedrons and \( \Omega'' \) is the collection of nonsimple polyhedrons. Then \( d = \max \{ \text{diam} \, (R(R)) : R \in \Omega \} \), \( m = \max \{ \mu_k[U T^\xi (R^*)] : \xi \in \Omega_n^k \} \), where \( U \) ranges over all \( R \in \Omega' \) with \( R \subset F^3 \). \( \sigma = \max \{ \sigma^\xi : \xi \in \Omega_n^k \} \), where \( \sigma^\xi = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \text{diam} \, T^\xi (B_{ij}) \), with \( \sum^* \).
Theorem (i). If \( k = 2 \) and \( V(T, F) < \infty \), then for every \( \epsilon > 0 \) there is a subdivision \( \mathcal{S} = \mathcal{S}' \cup \mathcal{S}'' \) of \( F \) with indices \( d, m, \sigma < \epsilon \).

Theorem (ii). If \( k \geq 2 \) and \( H_{n+1}^k[T(F)] = 0 \), then for every \( \epsilon > 0 \) there is a subdivision \( \mathcal{S} = \mathcal{S}' \cup \mathcal{S}'' \) of \( F \) with indices \( d, m, \sigma < \epsilon \).

Proof. The proofs of Theorems (i) and (ii) are essentially the same as the proof of \([3, (21.1.1)]\). The only change needed is the replacement of \([3, (16.10.i)]\) by (2.1.i) and (2.1.ii), respectively.

We observe that in the example of (2.2), \( d+m+\sigma > \epsilon_0 > 0 \) for an appropriate \( \epsilon_0 \), and \( H_{n+1}^k[T(F)] \neq 0 \).

4. Theorems on limits with respect to \( d, m, \sigma \). Let \((T, A) \in \mathfrak{F} (k, n)\), 

\( F_j \) (\( j = 1, 2, \cdots \)) be a sequence of figures with \( F_j \subset F_{j+1} \subset A \) and 

\( \bigcup_{j=1}^{\infty} F_j = A_0 \), and \( \mathcal{S}_j = \mathcal{S}_j' \cup \mathcal{S}_j'' \) be a subdivision of \( F_j \) with indices \( d_j, m_j, \sigma_j \) (\( j = 1, 2, \cdots \)).

Theorem (i). If \( A_0 \subset A \) is an admissible set and \( d_j + m_j + \sigma_j \to 0 \) as \( j \to \infty \), then

\[
\lim_{j \to \infty} \sum_{i} v(T, q) = V(T, A_0), \quad \lim_{j \to \infty} \sum_{i} v(T^i, q) = V(T^i, A_0), \quad \xi \in \Omega^k,
\]

where \( \sum_{i} \) ranges over all \( q \in \mathcal{S}_j', q \subset A_0 \).

If, furthermore, \( V(T, A) < \infty \), then

\[
\lim_{j \to \infty} \sum_{i} u(T, q) = U(T, A_0) = V(T, A_0),
\]

\[
\lim_{j \to \infty} \sum_{i} |u(T^i, q)| = U(T^i, A_0) = V(T^i, A_0), \quad \xi \in \Omega^k,
\]

where \( \sum_{i} \) ranges over all \( q \in \mathcal{S}_j', q \subset A_0 \).

Proof. The proof of (i) is essentially the same as \([3, (21.2.1), (21.3.1, ii)]\).

From (3.i, ii) we have sufficient conditions on \((T, A)\) for the existence of subdivisions \( \mathcal{S} = \mathcal{S}' \cup \mathcal{S}'' \) of \( F \subset A \) with arbitrary small indices \( d, m, \sigma \). Namely, (1) \( k = 2 \) and \( V(T, A) < \infty \); or (2) \( k \geq 2 \) and \( H_{n+1}^k[T(A)] = 0 \).

5. In this section we give some theorems concerning the Geöcze \( k \)-area of mappings \((T, A) \in \mathfrak{F} (k, n)\).

Theorem (i). If \((T, A) \in \mathfrak{F} (k, n) \) and \( H_{n+1}^k[T(A)] = 0 \), then
\[
\left[ \sum V^2(T^x, A) \right]^{1/2} \leq V(T, A) \leq \sum V(T^x, A), \quad \text{where } \sum \text{ ranges over } \xi \in \Omega^x_n.
\]

**Theorem (ii).** Let \((T, A) \in \mathfrak{I}(k, n)\) with \(H_{n+1}^* [T(A)] = 0\) and \(H \subset A\) be a set closed in \(A\) such that \(m_k [T^x(A^0 \cap H)] = 0, \xi \in \Omega^x_n\). Then each component \(A_i\) of \(A - H\) is an admissible set and \(V(T, A) = \sum V(T, A_i), V(T^x, A) = \sum V(T^x, A_i), \xi \in \Omega^x_n\), where \(\sum\) ranges over all components \(A_i\) of \(A - H\).

**Theorem (iii).** Let \((T, A) \in \mathfrak{I}(k, n)\) and \(H_{n+1}^* [T(A)] = 0\). Let \(A_i \subset A (i = 1, 2, \cdots)\) be a collection of sets open in \(A\) such that each \(A_i\) is the union of sets \(g \in \Gamma(T, A)\). Then we have \(V(T, A) \leq \sum V(T, A_i)\) and \(V(T^x, A) \leq \sum V(T^x, A_i), \xi \in \Omega^x_n\), where \(\cup\) and \(\sum\) range over all \(A_i\).

The proofs of the above theorems are essentially the same as [3, (21.4.i), (21.5.i), (25.1.iii)].

**Bibliography**