DOUBLY ITERATED MATRIX METHODS OF SUMMABILITY

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1. Introduction. A convenient generalization of the natural operator "limit" is realized in the concept of the summability-\( A \) of a sequence with respect to a matrix \( A \). For \( A = (a_{ik}) \), the sequence \( \{x_k\} \) is said to be summable-\( A \) to \( x \) provided

\[
x = \lim_{t \to \infty} \lim_{j \to \infty} \sum_{k=0}^{j} a_{ik}x_k.
\]

Here and throughout, elements of matrices and sequences are to be complex numbers. Indices run from 0 to \( \infty \); and in ambiguous cases the sequence index will be repeated as final subscript in the fashion \( \{x_{nk}\}_n \).

The matrix \( A \) is said to be regular if every convergent sequence is summable-\( A \) to its natural limit; for this the requirements on the \( a_{ik} \) are the celebrated Silverman-Toeplitz conditions [1, p. 64]. The idea of the present paper derives from the appeal to replace the application of the natural limit in (1) in both instances by summability-\( A \) itself, thereby yielding for regular \( A \) a not-less-general transform. More generally we consider the succession of functionals defined by repetitions of this double iteration.

Definition 1. With respect to a matrix \( A = (a_{ik}) \), the \( A^I \)-Operator of order \( n \), \( W_n \), for \( n = 0, 1, 2, \ldots \), is the functional, operating on sequences, defined by the following recursion:

\[
W_0\{x_k\} = \lim_{k \to \infty} x_k,
\]

and

\[
W_{n+1}\{x_k\} = W_n\{W_n\{\sum_{k=0}^{n} a_{ik}x_k\}\}\).
\]

Thus \( W_1\{x_k\} \) is the usual \( A \)-sum; and by way of example:

\[
W_2\{x_k\} = \lim_{p \to \infty} \sum_{i=0}^{p} a_{pi} \lim_{m \to \infty} \sum_{j=0}^{m} a_{mj} \sum_{k=0}^{j} a_{ik}x_k.
\]

Theorem 1. Let \( A \) be a regular matrix for which \( W_m \) and \( W_n \) are the respective \( A^I \)-Operators with \( m \geq n \). Then for every sequence \( \{c_k\} \) in the domain of definition of \( W_n \), we have

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\[ W_m \{ c_k \} = W_n \{ c_k \}. \]

**Proof.** We consider only the case \( m = n + 1 \); beyond that the proof is evident. For \( n = 0, m = 1 \), the conclusion is precisely the condition that \( A \) be regular, as hypothesized. Suppose the theorem is valid for \( n = p, m = p + 1 \). To complete the induction, establishing the case \( n = p + 1, m = p + 2 \), we must, according to Definition 1, show that

\[
W_{p+1} \left\{ \sum_{k=0}^{j} a_{ik}c_k \right\}_j = W_p \left\{ W_p \left\{ \sum_{k=0}^{j} a_{ik}c_k \right\}_j \right\}_j
\]

whenever the expression on the right is defined. But from the assumption of the theorem’s validity for \( n = p, m = p + 1 \), we have for each \( i \),

\[
W_{p+1} \left\{ \sum_{k=0}^{j} a_{ik}c_k \right\}_j = W_p \left\{ \sum_{k=0}^{j} a_{ik}c_k \right\}_j
\]

whenever the right side is defined. And similarly for the outside operators.

**Definition 2.** With respect to regular matrix \( A \), a sequence \( \{ c_k \} \) is said to be summable-\( AI \omega \) to \( c \), if for \( n \) sufficiently large we have \( W_n \{ c_k \} = c \).

**Note.** The relation \( \{ c_k \} \) summable-\( AI \omega \) to \( c \) may appropriately be written \( W_\omega \{ c_k \} = c \). Then as in Definition 1 we may define \( W_{\omega+1} \); and so on to general ordinal number index. But the problems raised by such generality present a distraction from the classical application that follows and so the subject of transfinite indices is deferred.

In the next section we consider the application of the \( AI \)-Operators to sequences of functions, extending the idea and useful properties of uniform convergence. In §3 we generalize a result of conventional summability-\( A \), establishing the effectiveness of the transforms \( W_n \) for summing Taylor Series in domains larger than the circle of convergence. Finally there is exhibited in §4 a simply derived matrix with respect to which the sequence of partial sums of each Taylor Series is summable-\( AI \omega \) to its analytic extension throughout the principal star domain of the function.

2. **Sequences of functions.** **Definition 3.** With respect to a matrix \( A = (a_{ik}) \), a sequence of functions \( \{ f_k(z) \} \) is said to be \( W_0 \)-uniform for \( z \) in a set \( T \) if \( \{ f_k(z) \} \) is uniformly convergent for \( z \in T \); and for \( n = 0, 1, 2, \ldots, \), \( \{ f_k(z) \} \) is said to be \( W_{n+1} \)-uniform for \( z \in T \) if for each \( i \), \( \{ \sum_{k=0}^{i} a_{ik}f_k(z) \}_j \) is \( W_n \)-uniform for \( z \in T \), and if in addition \( \{ W_n \{ \sum_{k=0}^{i} a_{ik}f_k(z) \}_j \}_i \) is defined and \( W_n \)-uniform for \( z \in T \).
It is immediate that "$W_n$-uniform" implies "$W_n$-summable." For conciseness we combine the concepts, writing simply: "$W_n\{f_k(z)\} = f(z)$ uniformly for $z \in T.$" It is likewise clear that if $W_n\{c_k\}$ is defined, then considering $\{c_k\}$ as a sequence of functions constant over a set $T$ it follows that $\{c_k\}$ is $W_n$-uniform for $z \in T.$ Here, as in the theorems following, the matrix defining $W_n$ is arbitrary, in particular it is not required to be regular.

**Theorem 2.** Suppose $W_n\{f_k(z)\} = f(z)$ and $W_n\{g_k(z)\} = g(z)$, both uniformly for $z \in T$, and let $h(z)$ be bounded for $z \in T$. Then

$$W_n\{f_k(z) + h(z) \cdot g_k(z)\} = f(z) + h(z) \cdot g(z)$$

uniformly for $z \in T$.

**Theorem 3.** Suppose each element of $\{f_k(z)\}$ is continuous for $z$ in a metric set $T$; suppose also that $\{f_k(z)\}$ is $W_n$-uniform for $z \in T$. Then $W_n\{f_k(z)\}$ is continuous for $z \in T$.

**Theorem 4.** Suppose $\{f_k(u, z)\}$ is $W_n$-uniform for $(u, z)$ in a set $C \times T$ where $C$ is a rectifiable contour of the complex plane. Further suppose that for each $k$ and each $z \in T$, $f_k(u, z)$ is continuous for $u \in C$. Then

$$W_n\{\int_C f_k(u, z) du\} = \int_C W_n\{f_k(u, z)\} du$$

uniformly for $z \in T$.

The proofs of these three theorems all conform to the same induction format. In each of them the case $n=0$ is commonplace, and the mechanics of passing from $m$ to $m+1$ is straightforward. We illustrate with the details of the proof of Theorem 4.

**Proof of Theorem 4.** As just observed the result for the case $n=0$, ordinary uniform convergence, is well known. Assume the theorem valid for $n=m$. Let $\{f_k(u, z)\}$ satisfy the hypotheses for the case $n=m+1$. First observe that

$$\sum_{k=0}^i a_{ik} \int_C f_k(u, z) du = \int_C \sum_{k=0}^i a_{ik} f_k(u, z) du.$$

From the assumed $W_{m+1}$-uniformity of $\{f_k(u, z)\}$, $\{\sum_{k=0}^i a_{ik} f_k(u, z)\}$ is, for each $i$, $W_m$-uniform for $(u, z) \in C \times T$. The continuity of $\sum_{k=0}^i a_{ik} f_k(u, z)$ for $u \in C$ follows from the continuity of the respective $f_k(u, z)$. Therefore from the $n=m$ case of the theorem we have for each $i$:
\[ W_m \left\{ \int_C \sum_{k=0}^{j} a_{ik} f_k(u, z) \, du \right\}_j = \int_C W_m \left\{ \sum_{k=0}^{j} a_{ik} f_k(u, z) \right\} \, du \]

uniformly for \( z \in T \).

Again by assumption \( \{ W_m \left\{ \sum_{k=0}^{j} a_{ik} f_k(u, z) \right\}_j \} \) is \( W_m \)-uniform for \((u, z) \in C \times T \). For each \( i \) and each \( z \in T \) the continuity of \( W_m \left\{ \sum_{k=0}^{j} a_{ik} f_k(u, z) \right\}_j \) follows from Theorem 3. Thus again from the \( n = m \) case of the theorem:

\[ W_m \left\{ \int_C W_m \left\{ \sum_{k=0}^{j} a_{ik} f_k(u, z) \right\}_j \right\} \, du \]

uniformly for \( z \in T \).

Collecting the steps we have

\[ W_m \left\{ W_m \left\{ \sum_{k=0}^{j} a_{ik} \int_C f_k(u, z) \, du \right\}_j \right\} \]

\[ = \int_C W_m \left\{ W_m \left\{ \sum_{k=0}^{j} a_{ik} f_k(u, z) \right\}_j \right\} \, du, \]

the sequences in \( j \) and \( i \) on the left being \( W_m \)-uniform for \( z \in T \). This is precisely the desired \( n = m + 1 \) result.

The linearity of the operators \( W_n \) we now observe as the all-functions-constant case of Theorem 2. The extension to the following statement is immediate.

**Theorem 5.** Each AI-Operator \( W_n \), as well as the summability-AI \( u \) transform for regular \( A \), defines a linear functional over a vector space of sequences of complex numbers.

3. **Application to Taylor series.** Applying an AI-Operator \( W_n \) to the partial sums of the Geometric Series, we have:

\[ W_n \left\{ \sum_{k=0}^{j} z^k \right\}_j = W_n \left\{ \frac{1 - z^{j+1}}{1 - z} \right\}_j = \frac{1}{1 - z} \left[ W_n \{1\}_j - z W_n \{z^j\}_j \right] \]

provided the expression on the right exists. (\( \{1\}_j \) represents the sequence of all 1’s). Thus sufficient conditions that \( W_n \) “properly” sum the Geometric Series at a point \( z \) are that \( W_n \{1\}_j = 1 \) and \( W_n \{z^j\}_j = 0 \). Theorem 6 provides an analogous result for Taylor Series in general.

A function of the form \( f(z) = \sum_{k=0}^{\infty} c_k z^k \) with positive radius of convergence will be regarded as extended to its principal star domain,
i.e., if there exists an analytic continuation of \( f(z) \) throughout a domain containing the segment \( \{ tz_0 \mid 0 \leq t \leq 1 \} \) then \( f(z_0) \) represents the value defined thereby. We represent the principal star domain as \( M_f \). Note that its complement \( \mathbb{C} \setminus M_f \) consists of those points which are singularities of the analytic function \( f(z) \) by a radial approach, automatically including all points "in the shadow of" such singularities.

In general a region \( Q \) is said to be starlike if for each \( z \in Q \) we have \( tz \in Q \) for \( 0 \leq t \leq 1 \). For a starlike domain \( Q \), the partial star domain \( P_{fQ} \) of \( f(z) \) with respect to \( Q \) is the intersection of the sets \( \xi Q \) as \( \xi \) ranges over \( \mathbb{C} \setminus M_f \). (\( \xi Q \) represents the set \( \{ \xi z \mid z \in Q \} \)).

The following result is an extension of a theorem of Okada [1, p. 189] applying for conventional matrix summability.

**Theorem 6.** Let \( f(z) = \sum_{k=0}^{\infty} c_k z^k \) have positive radius of convergence, and let \( s_j(z) = \sum_{k=0}^{\infty} c_k z^k \). Let \( W_n \) be an \( A_f \)-Operator with the properties that \( W_n \{ 1 \} = 1 \) and \( W_n \{ z^k \} = 0 \) uniformly for \( z \) in each closed and bounded set in a starlike domain \( Q \). Then \( W_n \{ s_j(z) \} = f(z) \) uniformly for \( z \) in each closed and bounded set in the partial star domain \( P_{fQ} \) of \( f(z) \) with respect to \( Q \).

**Proof.** Since \( Q \) clearly cannot contain the point \( z=1 \) it follows that \( P_{fQ} \) is a subset of \( M_f \). And since otherwise the theorem is vacuous we assume that \( Q \) contains the origin.

Let \( T \) represent a closed and bounded set in \( P_{fQ} \). For \( j = 0, 1, 2, \ldots \), consider the integral:
\[
I_j(z) = \frac{1}{2\pi i} \int_C \frac{f(u)}{z - u} \left( \frac{z}{u} \right)^{j+1} du.
\]

Here \( C \) is a rectifiable simple closed curve, taken counterclockwise, with the properties: \( f(u) \) is analytic on and inside \( C \); the origin and the set \( T \) are properly inside \( C \); and the union of all points of the form \( t/u \) for \( t \in T \) and \( u \in C \) forms a closed and bounded subset of \( Q \).

To verify the existence of such a contour \( C \) without a tedious direct construction we note that since \( M_f \) clearly satisfies the hypotheses of the Riemann Mapping Theorem there is an analytic function \( g(z) \) which simply maps the unit circle \( |z| < 1 \) onto \( M_f \). Since \( T \) is a closed and bounded set in \( M_f \) the pre-image of \( T \) under this mapping will be contained in a circle \( |z| < p_i < 1 \). Furthermore the definition of \( P_{fQ} \) insures that for all \( t \in T \subset P_{fQ} \) and \( \xi \in \mathbb{C} \setminus M_f \) we have \( (t/\xi) \in Q \). Thus if \( \gamma \in \mathbb{C} \) it follows that \( (t/\gamma) \in M_f \). Let \( S \) represent the union of the origin and all points of the form \( t/\gamma \) as \( t \) ranges over the closed and bounded set \( T \) and \( \gamma \) ranges over the closed and bounded-away-from-zero set \( \mathbb{C} \setminus M_f \). Clearly \( S \) is a closed and bounded set in \( M_f \). As with \( T \),

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the image of $S$ under the inverse mapping $g^{-1}(z)$ lies in a circle $|z| < \rho_2 < 1$. To recapitulate: if $t \in T$ implies $|g^{-1}(t)| < \rho_1 < 1$; and if $t \in T$ and $\rho_2 \leq |z| < 1$ implies $(t/g(z)) \in Q$. It follows immediately that the image under $g(z)$ of the circle $|z| = \max(\rho_1, \rho_2)$ furnishes an acceptable contour $C$.

The integrand in (2) is analytic inside $C$ except for poles at $u = 0$ and $u = z$. Observing that $I_j(0) = 0$ and $f(0) = s_j(0) = c_0$, we pass on to the case $z \neq 0$. The residue at $u = z$ is clearly $-f(z)$. Near $u = 0$ the integrand may be expanded thus:

$$\frac{z^{j+1}}{u^{j+1}} \frac{1}{z} \left( c_0 + c_1 u + \cdots + c_k u^k + \cdots \right) \left( 1 + \frac{u}{z} + \cdots + \frac{u^k}{z^k} + \cdots \right).$$

Collecting the coefficient of $u^{-1}$ we have the residue:

$$z^j \left( \frac{c_0}{z^j} + \frac{c_1}{z^{j-1}} + \cdots + c_j \right) = s_j(z).$$

It follows therefore that for all $z \in T$,

$$I_j(z) = s_j(z) - f(z).$$

From the hypotheses of the theorem, Theorems 2 and 4, and the properties of $C$, it is clear that

$$W_n\{I_j(z)\}_j = \frac{1}{2\pi i} \int_C \frac{f(u)}{z-u} W_n\left\{ \left( \frac{z}{u} \right)^j \right\} du = 0$$

uniformly for $z \in T$. Recalling Theorem 2 and the fact that $W_n\{1\}_k = 1$, we have finally

$$W_n\{I_j(z) + f(z)\}_j = f(z)$$

uniformly for $z \in T$. Since

$$I_j(z) + f(z) = s_j(z),$$

this is the desired result.

4. An example. Henceforth let $G(z) = (2 - z)^{-1}$, and let $A = (a_{ik})$ be the matrix whose elements are the coefficients of the Taylor Series of the functions $[G(z)]^i$. To be precise

$$[G(z)]^i = (2 - z)^{-i} = \sum_{k=0}^{\infty} a_{ik} z^k \quad \text{for} \quad |z| < 2.$$ 

Hence
This matrix \( A \), or more specifically the summability-\( A \) functional associated with it, represents the \( \alpha = 1/2 \) case of the \( S_\alpha \) methods which correspond to the family of matrices defined in the manner of (3) from the functions \( G_\alpha(z) = (1 - \alpha) / (1 - \alpha z) \) for \( 0 < \alpha < 1 \) \cite{2}. That \( A \) is regular follows immediately from the Silverman-Toeplitz Conditions. However we note that it is an easy exercise to demonstrate by an \((\varepsilon, N)\)-type argument that a given convergent sequence is summable-\( A \) to its natural limit. The rest of the paper is devoted to demonstrating the following result.

Theorem 7. Let \( W_n \) be the \( A-I \)-Operator of order \( n \) for the matrix \( A \) given by (4). Let \( f(z) = \sum_{k=0}^\infty c_k z^k \) have positive radius of convergence, and let \( s_j(z) = \sum_{k=0}^\infty c_k z^k \). Then for each closed and bounded set \( T \) in the principal star domain \( M_f \) of \( f(z) \), there is an \( N \) such that for all \( n > N \),

\[
W_n \{ s_j(z) \} ; = f(z)
\]

uniformly for \( z \in T \).

For a set \( T \) consisting of a single point, Theorem 7 takes the form:

Corollary. Let \( A, f(z) \) and \( s_j(z) \) be as in Theorem 7. Then \( \{ s_j(z) \} \) is summable-\( A-I \) to \( f(z) \) throughout the principal star domain of \( f(z) \).

In point of fact there have been exhibited matrices with respect to which the usual summability method (our \( W_i \)) sums \( \{ s_j(z) \} \) to \( f(z) \) throughout \( M_f \) \cite[pp. 181–187]{1}. But the construction of our matrix is very much simpler, and although summability-\( A-I \) arithmetically transcends ordinary summability-\( A \) it is nonetheless a perfectly natural generalization. Moreover the proof of Theorem 7 requires no more that an application on Theorem 6 together with a straightforward geometrical argument. The argument may clearly be adapted to a general class of matrices; in particular we note that Theorem 7 may be established in similar fashion for the matrices of all \( S_\alpha \) methods where \( 1/2 \leq \alpha < 1 \).

Proof of Theorem 7. Since the matrix \( A \) is regular it follows from Theorem 1 that \( W_n \{ 1 \}_k = 1 \) for all \( n \). Suppose for given \( n \), \( W_n \{ z^k \}_k = 0 \) uniformly in each closed set in a bounded starlike domain \( R_n \). Then applying Theorem 6 it follows that \( W_n \{ \sum_{k=0}^\infty a_{ik} z^k \}_j = [G(z)]^i \) for \( z \in 2R_n \), and \( W_n \{ [G(z)]^i \} ; = 0 \) for \( z \in G^{-1}(R_n) \), both with the usual uniformity in closed subsets. (Here, \( 2R_n = \{ z \mid (z/2) \in R_n \} \) and \( G^{-1}(R_n) = \{ z \mid G(z) \in R_n \} \). Thus if \( R_{n+1} \) is a bounded starlike domain contained in \( G^{-1}(R_n) \cap 2R_n \) then \( W_{n+1} \{ z^k \}_k = 0 \) uniformly in each closed set in \( R_{n+1} \).
With $G^0(z) = z$ and $G^{n+1}(z) = G(G^n(z))$, let $D_n$ represent the domain in which $|G^n(z)| < 1$. Thus $D_0$ is the unit circle; and with the usual linear fractional transformation manipulations it follows inductively that for $n = 1, 2, \cdots$, $D_n$ is the domain defined by the following inequality:

$$D_n: \left| z - \left(1 + \frac{1}{2n - 1}\right) \right| > \frac{1}{2n - 1}.$$

Let $D_n^*$ represent the starlike part of $D_n$, i.e., $D_n^*$ consists of those points $z$ of $D_n$ for which the segment $\{tz: 0 \leq t \leq 1\}$ does not intersect the circle bounding $D_n$. Finally let

$$Q_n = \bigcap_{k=0}^{n} 2^k D_{n-k}^*.$$

From the definition it follows that $Q_n$ is bounded and starlike, that $Q_{n+1} = D^*_{n+1} \cap 2Q_n$ and that $Q_{n+1} \supseteq Q_n$ with $\bigcup_{n=0}^{n} Q_n$ being precisely the complement of the segment $\{x: 1 \leq x\}$.

**Lemma.** With $W_n$ defined as in Theorem 7, $W_n \{z^k\}_k = 0$ uniformly for $z$ in each closed set in the domain $Q_n$ of (6).

**Proof of Lemma.** Since $Q_0$ is the unit circle, the $n = 0$ (natural convergence) case of the lemma is immediate. To deduce the validity of the lemma for $n + 1$ assuming its validity for $n$, we employ the argument spelled out above in terms of domains $R_n$ and $R_{n+1}$. That $Q_n$ is bounded and starlike has already been observed, so we have only to demonstrate that $Q_{n+1} \subseteq G^{-1}(Q_n) \cap 2Q_n$. But as has also been observed, $Q_{n+1} = D^*_{n+1} \cap 2Q_n$, so it suffices to show that $D^*_{n+1} \subseteq G^{-1}(Q_n)$. Referring to (6) the proof reduces to showing that $D^*_{n+1} \subseteq G^{-1}(2^k D^*_{n-k})$ for $k = 0, 1, \cdots, n$.

The character of the domains $D^*_{n+1}$, $2^k D^*_{n-k}$ and $G^{-1}(2^k D^*_{n-k})$, for $k = 0, 1, \cdots, n - 1$, is indicated in Figure 1. The former two are plotted directly from definition; the construction of $G^{-1}(2^k D^*_{n-k})$ for the transformation $G^{-1}(z) = 2 - z^{-1}$ is immediate. From the geometry it is clear that the condition $D^*_{n+1} \subseteq G^{-1}(2^k D^*_{n-k})$ is equivalent to the following inequality comparing the ratio of diameter to distance from the origin for the circles generating $D^*_{n+1}$ and $G^{-1}(2^k D^*_{n-k})$ respectively:

$$\frac{2}{2n + 1} \geq \frac{1}{2 - 2^k} \left[ \left(2 - \left(2^k \left(1 + \frac{2}{2(n - k) - 1}\right)\right)^{-1}\right) - (2 - 2^k) \right].$$
In the exceptional case $k = n$, the region $2^k D_{n-k}^*$ is simply the circle $|z| < 2^n$, and $G^{-1}(2^k D_{n-k}^*)$ is the domain $|z-2| > 2^{-n}$. Reasoning as before we find that the condition $D_{n+1}^* \subseteq G^{-1}(2^k D_{n-k}^*)$ for $k = n$ is indeed equivalent to the $k = n$ case of the inequality (7). Thus the proof of the lemma has reduced to proving the inequality (7) for $n = 0, 1, 2, \cdots$ with $k = 0, 1, \cdots, n$. Simplifying (7), we obtain the following equivalent forms:
\[
\frac{2}{2n + 1} \geq \frac{1}{2^{k+1} - 1} \left[ 1 - \frac{2(n - k) - 1}{2(n - k) + 1} \right] = \frac{2}{(2^{k+1} - 1)(2(n - k) + 1)},
\]

\((2^{k+1} - 1)(2(n - k) + 1) \geq 2n + 1,
\]

\((2^{k+1} - 2)(2(n - k) + (2^{k+1} - 2k - 2)) \geq 0.\]

Since in the last form both terms on the left are clearly non-negative for admissible values of \(n\) and \(k\), the desired inequality is established. This completes the proof of the lemma.

Returning to the proof of the theorem proper, recall that for the domains \(Q_n\) of (6), \(Q_{n+1} \supset Q_n\) and \(\bigcup_{n=0}^\infty Q_n\) omits only the half-line \([1, \infty]\). Since the principal star domain \(M_f\) is starlike, \(z \in M_f\) and \(\xi \in \partial M_f\) implies \((z/\xi) \in [1, \infty]\). Thus for a closed and bounded set \(T\) contained in \(M_f\), the union of the origin and all points of the form \(z/\xi\) for \(z \in T\) and \(\xi \in \partial M_f\) forms a closed and bounded set \(V\) not intersecting \([1, \infty]\). Clearly for \(n\) sufficiently large, \(V \subset Q_n\). But for \(z \in T; \xi \in \partial M_f\) implies \((z/\xi) \in V \subset Q_n\), or in other words, \(z \in \partial Q_n\); and thus \(z \in P_{fQ_n}\). Therefore \(T\) is a closed and bounded set in \(P_{fQ_n}\). Recalling the lemma and the fact that \(W_n\{1\}_k = 1\), and applying Theorem 6, the conclusion of Theorem 7 follows.

References


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