CARTESIAN PRODUCTS WITH INTERVALS

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1. Introduction.

THEOREM. *There exists a combinatorial n-manifold M^n with boundary B such that $M^n \times I = I^{n+1}$ and $\pi_1(B) \neq 1$ if and only if $n \geqq 4$.*

It has been proved by Bing [1] that such manifolds cannot exist for $n<4$. For $n=4$ an example has been given by Poénaru [6], and we give here a construction which gives such an M^n for all $n \geqq 5$. The proof uses the affirmative solution of the generalized Poincaré Conjecture (for $n \geqq 7$ by Stallings [7] and for $n=5$, 6 by Zeeman [10]), and the generalized Schoenflies theorem.

2. Two consequences of the Stallings-Zeeman results.[2]

PROPOSITION 1. *If M^n is a contractible combinatorial n-manifold, $n \geqq 5$, and the boundary of M^n is S^{n-1}, then M^n is homeomorphic with I^n.*

PROOF. Attach two copies of M^n along S^{n-1}, using the identity map. The result is a combinatorial n-manifold T^n, and using van Kampen's theorem and duality we get that T^n has the homotopy type of S^n. Since $n \geqq 5$, T^n is homeomorphic with S^n, and we note that S^{n-1} is nicely embedded. By the Schoenflies theorem ([4] or [2]), it follows that M^n is an n-cube.

PROPOSITION 2. *If M^n is a contractible combinatorial n-manifold and $n \geqq 5$, then $M^n \times I$ is homeomorphic with I^{n+1}.*

PROOF. Let B be the boundary of M^n. Then the boundary of $M^n \times I$ is $T^n = (M^n \times \{0, 1\}) \cup (B \times I)$. Again it is routine to check that T^n has the homotopy type of S^n, so we conclude that T^n is an n-sphere. By Proposition 1 we have that $M^n \times I$ is homeomorphic with I^{n+1}.

3. Construction. Let P be the 2-polyhedron defined by Newman in [5]. This polyhedron has $\pi_1(P) \neq 1$ whereas $H_1(P, Z) = 0 = H_2(P, Z)$. For $n \geqq 5$ we embed P in S^n as a subcomplex and use Newman's result that $\pi_1(S^n - P) = 1$. Let T be a regular neighborhood of P with P lying in the interior of T (see [8, p. 293]). Then the boundary B of

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[2] These two propositions have also been noted by J. J. Andrews, *Imbedding homotopy cells*, to appear.

T is an $(n-1)$-manifold which is also the boundary of $S^n - \text{Int. } T$. We define $M^n = S^n - \text{Int. } T$.

Now the natural deformation retraction of T onto P induces a map $\phi: B \rightarrow P$ and it is known that $\phi: \pi_1(B) \rightarrow \pi_1(P)$ is surjective [**3**; **5**]. Hence $\pi_1(B) \neq 1$, and to complete our construction we need only to show that M^n is contractible and apply Proposition 2.

Now $\pi_1(M^n) = \pi_1(S^n - P)$ since $S^n - P$ may be deformation retracted onto M^n, and it follows that $\pi_1(M^n) = 1$. Since T has the same homotopy type as P, it is homologically trivial. By duality, M^n is homologically trivial and hence contractible.

References

1. R. H. Bing, *A 3-cell is the only object whose cartesian product with an arc is a 4-cell*, Abstract 564-257, Notices Amer. Math. Soc. vol. 7 (1960) p. 68.

2. Morton Brown, *A proof of the generalized Schoenflies theorem*, Bull. Amer. Math. Soc. vol. 66 (1960) pp. 74–76.

3. M. L. Curtis and R. L. Wilder, *The existence of certain types of manifolds*, Trans. Math. Soc. vol. 91 (1959) pp. 152–160.

4. Barry Mazur, *On embeddings of spheres*, Bull. Amer. Math. Soc. vol. 65 (1959) pp. 59–65.

5. M. H. A. Newman, *Boundaries of ULC sets in euclidean n-space*, Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) pp. 193–196.

6. V. Poénaru, *La décomposition de l'hypercube en produit topologique*, Bull. Soc. Math. France vol. 88 (1960) pp. 113–129.

7. John Stallings, *Polyhedral homotopy-spheres*, Bull. Amer. Math. Soc. vol. 66 (1960) pp. 485–488.

8. J. H. C. Whitehead, *Simplicial spaces, nuclei and m-groups*, Proc. London Math. Soc. vol. 45 (1939) pp. 243–327.

9. R. L. Wilder, *On free subsets of E^n*, Fund. Math. vol. 21 (1933) pp. 156–167.

10. E. C. Zeeman, *The generalized Poincaré conjecture*, Bull. Amer. Math. Soc. vol. 67 (1961) p. 270.

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