A REMARK ON THE RITT ORDER OF ENTIRE FUNCTIONS DEFINED BY DIRICHLET SERIES

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1. Let \( f(s) = \sigma + it \) be an entire function defined by an everywhere absolutely convergent Dirichlet series

\[
\sum_{n=1}^{\infty} a_n \exp(\lambda_n s) \quad (0 < \lambda_n < \lambda_{n+1} \to \infty).
\]

The Ritt order is defined \([1, \text{p. 77}]\) as

\[
\rho = \lim \sup_{\sigma \to \infty} [\log \log M(\sigma)/\sigma]
\]

with

\[
M(\sigma) = \sup_{-\infty < t < \infty} |f(\sigma + it)|.
\]

2. We will show that the Ritt theorem \([1, \text{p. 78}]\), which gives the value of \( \rho \) in terms of the sequences \( \{|a_n|\} \) and \( \{\lambda_n\} \) under the restriction \( \lim \inf_{n \to \infty} [\lambda_n / \log n] > 0 \), can be improved as follows:

Let \( \rho_1 \) be defined as

\[
\rho_1 = \lim \sup_{n \to \infty} [\lambda_n \log \lambda_n / \log |a_n|^{-1}].
\]

We will prove the following:

**Theorem.** If \( \sum_{n=1}^{\infty} a_n \exp(\lambda_n s) \) is absolutely convergent for all \( s \), then \( \rho \geq \rho_1 \), and if, in addition, it is assumed that

\[
(1) \quad \lim_{n \to \infty} [\lambda_n \log \lambda_n / \log n] = \infty
\]

then \( \rho = \rho_1 \).

**Proof.** With no changes, the first part of the proof given by Mandelbrojt \([2, \text{p. 217}]\) shows that \( \rho \geq \rho_1 \) if \( \rho < \infty \), and the same conclusion obviously holds if \( \rho = \infty \).

To completely establish the theorem it is now enough to prove that if \( (1) \) holds, then \( \rho \leq \rho_1 \). This result can be obtained by a modification of the second part of Mandelbrojt’s proof \([2, \text{p. 218}]\), as follows.

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We can assume $\rho_1 < \infty$, since otherwise the conclusion is trivial. Then for every $\delta > 0$ there exists $n_0 = n_0(\delta)$ such that $|a_n| \leq \exp \left[ -\lambda_n \log \lambda_n/(\rho_1 + \delta) \right]$ for all $n \geq n_0$. Therefore, for every $\sigma$

$$M(\sigma) \leq \sum_{n=1}^{n_0-1} |a_n| \exp(\lambda_n \sigma)$$

$$+ \sum_{n=n_0}^{\infty} \exp[-\lambda_n \log \lambda_n/(\rho_1 + \delta) + \lambda_n \sigma]$$

because the series involved in the formula is easily seen to be convergent, since from (1) it follows that for every $\varepsilon > 0$, the series $\sum_{n=1}^{\infty} \exp (-\varepsilon \lambda_n \log \lambda_n)$ converges to a finite sum $\phi(\varepsilon)$.

Now, for any choice of $\alpha \geq 0$ and $\beta > 0$ such that $\alpha + \beta = 1/(\rho + \delta)$ we have

$$M(\sigma) \leq \sum_{n=1}^{n_0-1} |a_n| \exp(\lambda_n \sigma) + \max_{n \geq 1} \left[ \exp(-\alpha \lambda_n \log \lambda_n + \lambda_n \sigma) \right]$$

$$\cdot \sum_{n=n_0}^{\infty} \exp(-\beta \lambda_n \log \lambda_n)$$

$$\leq \sum_{n=1}^{n_0-1} |a_n| \exp(\lambda_n \sigma) + \phi(\beta) \max_{x \geq 0} \exp(-ax \log x + x \sigma)$$

$$= \sum_{n=1}^{n_0-1} |a_n| \exp(\lambda_n \sigma) + \phi(\beta) \exp\left\{ \alpha \exp\left[ (\sigma - \alpha)/\alpha \right] \right\}$$

and consequently $\rho = \limsup_{r \to \infty} \left[ \log \log M(\sigma)/\sigma \right] \leq 1/\alpha$, and since $\sup \alpha = 1/(\rho_1 + \delta)$ and $\delta$ is arbitrarily small, it follows that $\rho \leq \rho_1$ and the theorem is proved.

Finally, it should be noticed that the assumption of the sequence $\{\lambda_n\}$ being monotonic, implies that (1) holds if and only if the series $\sum \exp (-\varepsilon \lambda_n \log \lambda_n)$ converges for every $\varepsilon > 0$, and therefore the previous theorem is the best possible result that can be obtained by the present method of proof.

**References**


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