THE HAAR PROBLEM IN $L_1$

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Let $B$ be a real Banach space and $E = \{x_1, x_2, \ldots, x_n\}$ a finite subset of $B$. Let $D$ be the linear manifold spanned by $E$. For every $x \in B$ the distance of $x$ from $D$ is attained at some member of $D$, i.e., there exist scalars $[\alpha_1, \alpha_2, \ldots, \alpha_n]$ such that

$$\|x - \sum \alpha_i x_i\| = \inf_{y \in D} \|x - y\|.$$ 

$E$ is said to have the Haar Property if the $\alpha$'s in (1) are uniquely determined for every $x \in B$. The Haar Problem consists of finding a necessary and sufficient condition on $E$ in order that $E$ have the Haar property.

Haar [1] solved the above problem for the case where $B$ is the collection of real-valued continuous functions on a compact subset of Euclidean $n$-space under the sup norm. It is easy to see [2] that if $B$ is strictly-normed, i.e., if

$$(\|x + y\| = \|x\| + \|y\|) \Rightarrow y = sx$$

for a scalar $s$ then every linearly-independent set $E$ enjoys the Haar property. Other results have also been obtained [3; 4].

The purpose of this note is to treat the case where $B$ is $L_1(M)$, the collection of real-valued integrable functions on a finite nonatomic measure space $M$ under the $L_1$ norm. (For $L_p(M)$ with $p > 1$ (2) holds.) The following theorem will be proved:

**Theorem.** No finite subset of $L_1(M)$ has the Haar property.

The proof depends on the following theorem of Liapounoff and Halmos [5; 6]: The range of a countably-additive, finite, nonatomic measure with values in a real finite-dimensional vector space is convex.

**Lemma.** Let $f_1, f_2, \ldots, f_n$ be $n$ functions in $L_1(M)$, $M$ a finite nonatomic measure space. There exists a measurable subset $E$ of $M$ such that $\phi = \chi_E - \chi_{E'}$ satisfies

$$\int \phi f_i = 0 \quad (i = 1, 2, \ldots, n).$$

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1 The author is indebted to G.-C. Rota, who introduced him to this problem, and A. Browder, who called his attention to the result of Liapounoff mentioned in the text.

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Proof. Assume first that each $f_i$ is either strictly positive, strictly negative, or zero, a.e. in $M$. Eliminate the zero functions (since for these any $\phi$ will do), and define a vector-valued measure on the measurable subsets of $M$ by

$$\lambda(E) = \left( a_1 \int \chi_E f_1, a_2 \int \chi_E f_2, \cdots, a_n \int \chi_E f_n \right)$$

where $a_i = (f_i)^{-1}$. Since $M$ is nonatomic, each coordinate of $\lambda$ is nonatomic. By Liapounoff's theorem the range of $\lambda$ is convex. Since $\lambda(M) = (1, 1, \cdots, 1)$ and $\lambda(0) = (0, 0, \cdots, 0)$, there is a set $E$ in $M$ such that $\lambda(E) = (1/2, 1/2, \cdots, 1/2)$. This $E$ is the $E$ of the lemma.

In the more general case $M$ may be partitioned into a finite disjoint union of measurable sets $M_j$ on any one of which each $f_i$ is strictly positive, strictly negative, or zero. On each $M_j$ the above argument supplies an $E_j$, and the union of these $E_j$ is the $E$ of the lemma.

Proof of the Theorem. Let $f_1, f_2, \cdots, f_n$ be $n$ members of $L_1(M)$. With $\phi$ as in the lemma, define a function $g$ in $L_1(M)$ by $g = \phi[|f_1| + |f_2| + \cdots + |f_n|]$. For any scalars $\{\alpha_i\}$

$$\|g - \sum \alpha_i f_i\| = \int |g - \sum| \geq \int \phi(g - \sum) = \int \phi g = \|g\|.$$ 

On the other hand, for every set of scalars $\{\alpha_i\}$ there is an $\epsilon > 0$ such that $|\epsilon \sum \alpha_i f_i| < |g|$ hence

$$\|g - \epsilon \sum \alpha_i f_i\| = \int |g - \epsilon \sum| = \int \phi(g - \epsilon \sum) = \|g\|.$$ 

Thus for $g$ the $\alpha_i$ in (1) are not only not unique—the admissible vectors $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ contain a sphere about the origin.

Added in Proof: The author has learned that the result given here had already appeared as Theorem 2.5 in [7].

References

THE GEÖCZE $k$-AREA AND A CYLINDRICAL PROPERTY

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In [5], a definition of the Geöcze $k$-area of a mapping from admissible sets of Euclidean $k$-space $E_k$ into Euclidean $n$-space $E_n$ $(2 \leq k \leq n)$ is given. This definition is an extension of the Geöcze area given in [3]. With this definition of Geöcze $k$-area, a treatment of Geöcze $k$-area is developed for flat mappings ($k = n$) paralleling the treatment of Geöcze area for plane mappings given in [3]. The present paper gives results concerning the Geöcze $k$-area for mappings from admissible sets of $E_k$ into $E_n$ $(n > k \geq 2)$. A cylindrical property is defined for mappings in harmony with [3, (16.10)]. This property, which has had an essential part in the proofs of the main theorems for Lebesgue area for mappings from admissible sets of $E_k$ into $E_n$ [3] and which has been used in other research, is shown to play a prominent role in the extension of the theory of Geöcze area to higher dimensions. An example is given to show that the theorems concerning the cylindrical property in [3] are no longer valid for $k \geq 3$. These theorems are shown to be valid under a certain restrictive hypothesis found in the literature.

1. Notations and definitions. If $X$ is a set in $E_k$, then $\overline{X}$, $X^0$, and $X^*$ will denote respectively the closure, interior, and boundary of $X$.

A polyhedral region $R$ in $E_k$ is the point-set covered by a strongly connected $k$-complex situated in $E_k$. A polyhedral region $R$ is called simple if $E_k - R$ is connected (see [5]).

By a figure $F$ we mean a finite union of nonoverlapping polyhedral regions in $E_k$ such that the interior of the union is the union of the interior of the finitely many polyhedral regions. A set $A$ in $E_k$ is said to be admissible in each of the following cases:

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