THE HAAR PROBLEM IN $L_1$

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Let $B$ be a real Banach space and $E = [x_1, x_2, \ldots, x_n]$ a finite subset of $B$. Let $D$ be the linear manifold spanned by $E$. For every $x \in B$ the distance of $x$ from $D$ is attained at some member of $D$, i.e., there exist scalars $[\alpha_1, \alpha_2, \ldots, \alpha_n]$ such that

$$\left\| x - \sum_i \alpha_i x_i \right\| = \inf_{y \in D} \| x - y \|.$$  

$E$ is said to have the Haar Property if the $\alpha$'s in (1) are uniquely determined for every $x \in B$. The Haar Problem consists of finding a necessary and sufficient condition on $E$ in order that $E$ have the Haar property.

Haar [1] solved the above problem for the case where $B$ is the collection of real-valued continuous functions on a compact subset of Euclidean $n$-space under the sup norm. It is easy to see [2] that if $B$ is strictly-normed, i.e., if

$$||x + y|| = ||x|| + ||y|| \Rightarrow y = sx$$

for a scalar $s$ then every linearly-independent set $E$ enjoys the Haar property. Other results have also been obtained [3; 4].

The purpose of this note is to treat the case where $B$ is $L_1(M)$, the collection of real-valued integrable functions on a finite nonatomic measure space $M$ under the $L_1$ norm. (For $L_p(M)$ with $p > 1$ (2) holds.) The following theorem will be proved:

**Theorem.** No finite subset of $L_1(M)$ has the Haar property.

The proof depends on the following theorem of Liapounoff and Halmos [5; 6]: The range of a countably-additive, finite, nonatomic measure with values in a real finite-dimensional vector space is convex.

**Lemma.** Let $f_1, f_2, \ldots, f_n$ be $n$ functions in $L_1(M)$, $M$ a finite nonatomic measure space. There exists a measurable subset $E$ of $M$ such that

$$\int \phi = 0 \quad (i = 1, 2, \ldots, n).$$

Received by the editors November 3, 1960.

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Proof. Assume first that each $f_i$ is either strictly positive, strictly negative, or zero, a.e. in $M$. Eliminate the zero functions (since for these any $\phi$ will do), and define a vector-valued measure on the measurable subsets of $M$ by

$$
\lambda(E) = \left( a_1 \int \chi_E f_1, a_2 \int \chi_E f_2, \ldots, a_n \int \chi_E f_n \right)
$$

where $a_i = (\langle f_i \rangle)^{-1}$. Since $M$ is nonatomic, each coordinate of $\lambda$ is nonatomic. By Liapounoff's theorem the range of $\lambda$ is convex. Since $\lambda(M) = (1, 1, \ldots, 1)$ and $\lambda(0) = (0, 0, \ldots, 0)$, there is a set $E$ in $M$ such that $\lambda(E) = (1/2, 1/2, \ldots, 1/2)$. This $E$ is the $E$ of the lemma.

In the more general case $M$ may be partitioned into a finite disjoint union of measurable sets $M_j$ on any one of which each $f_i$ is strictly positive, strictly negative, or zero. On each $M_j$ the above argument supplies an $E_j$, and the union of these $E_j$ is the $E$ of the lemma.

Proof of the Theorem. Let $f_1, f_2, \ldots, f_n$ be $n$ members of $L_1(M)$. With $\phi$ as in the lemma, define a function $g$ in $L_1(M)$ by $g = \phi[|f_1| + |f_2| + \cdots + |f_n|]$. For any scalars $\{\alpha_i\}$

$$
\|g - \sum \alpha_i f_i\| = \int |g - \sum| \geq \int \phi(g - \sum) = \int \phi g = \|g\|.
$$

On the other hand, for every set of scalars $\{\alpha_i\}$ there is an $\epsilon > 0$ such that $|\epsilon \sum \alpha_i f_i| < |g|$ hence

$$
\|g - \epsilon \sum \alpha_i f_i\| = \int |g - \epsilon \sum| = \int \phi(g - \epsilon \sum) = \|g\|.
$$

Thus for $g$ the $\alpha_i$ in (1) are not only not unique—the admissible vectors $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ contain a sphere about the origin.

Added in Proof: The author has learned that the result given here had already appeared as Theorem 2.5 in [7].

References

THE GEÖCZE \( k \)-AREA AND A CYLINDRICAL PROPERTY

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In [5], a definition of the Geöcze \( k \)-area of a mapping from admissible sets of Euclidean \( k \)-space \( E_k \) into Euclidean \( n \)-space \( E_n \) \( (2 \leq k \leq n) \) is given. This definition is an extension of the Geöcze area given in [3]. With this definition of Geöcze \( k \)-area, a treatment of Geöcze \( k \)-area is developed for flat mappings \( (k = n) \) paralleling the treatment of Geöcze area for plane mappings given in [3]. The present paper gives results concerning the Geöcze \( k \)-area for mappings from admissible sets of \( E_k \) into \( E_n \) \( (n > k \geq 2) \). A cylindrical property is defined for mappings in harmony with [3, (16.10)]. This property, which has had an essential part in the proofs of the main theorems for Lebesgue area for mappings from admissible sets of \( E_k \) into \( E_n \) [3] and which has been used in other research, is shown to play a prominent role in the extension of the theory of Geöcze area to higher dimensions. An example is given to show that the theorems concerning the cylindrical property in [3] are no longer valid for \( k \geq 3 \). These theorems are shown to be valid under a certain restrictive hypothesis found in the literature.

1. Notations and definitions. If \( X \) is a set in \( E_k \), then \( \overline{X}, X^0, \) and \( X^* \) will denote respectively the closure, interior, and boundary of \( X \).

A polyhedral region \( R \) in \( E_k \) is the point-set covered by a strongly connected \( k \)-complex situated in \( E_k \). A polyhedral region \( R \) is called simple if \( E_k - R \) is connected (see [5]).

By a figure \( F \) we mean a finite union of nonoverlapping polyhedral regions in \( E_k \) such that the interior of the union is the union of the interior of the finitely many polyhedral regions. A set \( A \) in \( E_k \) is said to be admissible in each of the following cases:

Presented to the Society, January 29, 1960 under the title The Geöcze \( k \)-area; received by the editors September 21, 1960.