LEBESGUE SPACES OF SUMMABLE FUNCTIONS[1]

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1. Introduction. The classical theorem of Kakutani [1] provides an elegant characterization of L^1 in terms of its lattice order and special properties of the norm. Much of the difficulty in representing an abstract (L)-space apparently stems from the fact that the end result must in general be a vector lattice whose elements are Lebesgue classes of functions rather than functions themselves. In this paper we give a characterization of $\mathfrak{L}^1$, the space of summable functions. Toward this end, we abstract the essential properties of $\mathfrak{L}^1$ in the following:

DEFINITION. An $(\mathfrak{A}\mathfrak{L})$-*space* is a vector lattice V with a seminorm p satisfying:

(1) $p(x+y)=p(x)+p(y)$ for $x, y \geqq 0$.

(2) $p(|x|)=p(x)$ for all $x \in V$.

(3) V is complete in the p-topology.

(4) There is a total family Λ of linear lattice functionals (see §2 for definition) such that the subspace

$$B = \left\{ x \in V : \|x\| = \sup_{\lambda \in \Lambda} |\lambda(x)| < \infty \right\}$$

is dense in V under the p-topology, and complete under the norm topology given by $\|x\|$.

MAIN THEOREM. *Let V be an $(\mathfrak{A}\mathfrak{L})$-space. Then there is a locally compact Hausdorff space E and a unique positive Radon measure μ on E such that V is linearly, latticially and isometrically isomorphic to $\mathfrak{L}^1(E, \mu)$, the space of all summable functions on E. Of course $V/p^{-1}(0)$ is then abstractly identical with $L^1(E, \mu)$.*

For example, if we take V to be the space $\mathfrak{L}^1(X, m)$ of summable functions on a (nontopological) finite measure space (X, S, m) then conditions (1)–(3) are clearly satisfied. For Λ we may take the "point measures" (linear lattice functionals) ϵ_a, $a \in X$ where $\epsilon_a(f)=f(a)$, for f in $\mathfrak{L}^1$. B is then the Banach algebra of bounded summable functions and (4) is clearly satisfied.

If we strengthen the above conditions by requiring p to be a norm so that V becomes an (AL)-space in Kakutani's sense, then it appears likely that the only such spaces are the l^1 spaces of summable func-

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tions over a discrete space relative to an atomic measure. This question will be considered in a later paper.

2. **The auxiliary representation.** By a *linear lattice functional* we mean a linear functional λ: $V \to R$ satisfying $\lambda(x^+) = (\lambda x)^+$, for all $x \in V$, where $x^+ = x \vee 0$. Such a functional is evidently positive. Regarding the existence of such functionals we have the following result (Nakayama [**3**]):

THEOREM 1. *Let V be a vector lattice. Then there exists a set X such that V is isomorphic, as a vector lattice, to a pointwise linear sublattice of R^X if and only if V has a total family of linear lattice functionals.*

Thus condition (4) amounts, in part, to assuming that V is a function lattice. Regarding (4) we observe further that the set B inherits some rather special properties. For a set Λ, we denote by $m(\Lambda)$ the algebra of all bounded real-valued functions on Λ.

LEMMA 1. *Let V be a pointwise vector lattice of functions on a set Λ and let $B = m(\Lambda) \cap V$. Then B is a lattice ideal in V, and, a fortiori, an algebra.*

PROOF. Let $f \in m(\Lambda)$ with $\|f\| > 1$ where $\|f\| = \sup\{|f(\lambda)| : \lambda \in \Lambda\}$. Then $\|f\| \leqq \|f^2\|$; for if we take $\lambda \in \Lambda$ so that $|f(\lambda)| > 1$, then $|f(\lambda)| < |f^2(\lambda)|$. Again, if $f \in m(\Lambda)$, $\|f\| > 1$, then $f^2 \leqq \|f^2\| \cdot |f|$; for we have $\|f^2\| \cdot |f| - |f|^2 = (\|f^2\| - |f|) \cdot |f| \geqq 0$ since $|f| \leqq \|f\| \leqq \|f^2\|$. Now if $g \in B$ and $f \in V$ with $|f| \leqq |g|$, then $\|f\| \leqq \|g\| < \infty$, so $f \in B$, i.e., B is a lattice ideal in V. For $0 \neq f \in B$ we choose $0 < \epsilon < \|f\|$. Then $g = (1/\epsilon)f$ has norm > 1. Hence $0 \leqq g^2 \leqq \|g^2\| \cdot |g| = (\|g^2\|/\epsilon) \cdot |f|$ and by the lattice ideal property $g^2 \in B$. But $f^2 = \epsilon^2 g^2$, so $f^2 \in B$. Q.E.D.

Reverting now to Theorem 1, we map V into R^Λ by $x \to \bar{x}$ where $\bar{x}(\lambda) = \lambda(x)$ for $x \in V$, $\lambda \in \Lambda$. The elements of V then appear as functions on Λ; consequently we shall identify V and B with their images under this representation and define the uniform norm on B by: $\|f\| = \sup\{|\bar{f}(\lambda)| : \lambda \in \Lambda\} = \sup\{|\lambda(f)| : \lambda \in \Lambda\}$ for $f \in B$. Condition (4) assures us that $\|f\|$ is finite and that B is sufficiently large. It should be remarked that this representation is only a vehicle and will be discarded as soon as the desired function space appears.

3. **The integral.** We now focus our attention on conditions (1) and (2). Let $V^+ = \{x \in V : x \geqq 0\}$.

LEMMA 2. *Let ν: $V^+ \to R$ be a positive additive functional. Then there is a unique positive linear functional μ: $V \to R$ agreeing with ν on V^+.*

PROOF. See Bourbaki [**4**, p. 34, Proposition 2].

LEMMA 3. *Let V be a vector lattice with a semi-norm p. Then* (1) *and* (2) *hold if and only if there is a positive linear functional μ on V such that $p(x)=\mu(|x|)$, for all $x\in V$. Moreover, if* (1) *and* (2) *hold, μ is uniquely determined.*

PROOF. *Sufficiency*. In any vector lattice we have the identities:

$$|x+y| = |x| + |y| \qquad \text{for } x, y \geqq 0. \tag{1$'$}$$

$$|(|x|)| = |x| \qquad \text{for all } x \in V. \tag{2$'$}$$

The additivity of μ together with the assumption $p(x)=\mu(|x|)$ easily imply (1) and (2).

Necessity and uniqueness. Condition (1) says that p is additive on the positive cone V^+. As in the proof of Lemma 2, we define $\mu(x)=p(x^+)-p(x^-)$. Clearly $p(x)=p(|x|)=\mu(|x|)$. Q.E.D.

4. **Representation of** B. The fact that B is a pointwise algebra of bounded functions (Lemma 1) enables us to adjoin an order unit e (the constant function 1) to B (for definition of an order unit see Kadison [**5**, p. 3]). Thus for any function f in B we can find an $\alpha\geqq 0$ such that $|f|\leqq \alpha e$. Let B_e denote the algebra resulting from the adjunction of e to B. Being a function lattice, B_e is manifestly Archimedean in Kadison's sense and the "natural norm" $\|f\|=\inf\{\alpha: |f|\leqq \alpha e\}$ is just the uniform norm inherited from $m(\Lambda)$. By condition (4), B is complete in the uniform norm as is B_e.

We now appeal to a classical representation theorem for vector lattices.

THEOREM 2. *Let L be an Archimedean vector lattice with order unit e. If L is complete in the norm $\|f\|=\inf\{\alpha: |f|\leqq \alpha e\}$ then one can find a compact Hausdorff space S and a linear lattice isometry of L with $C(S)$.*

For the proof, we refer to [**2**, p. 103, Theorem 3] or to [**5**, p. 10, Theorem 4.1 and Historical Remarks].

COROLLARY. *B is linearly, latticially and isometrically isomorphic to $C_\infty(E)$, the Banach algebra of all continuous functions on $E=S-\{\infty\}$ (locally compact Hausdorff) vanishing at ∞.*

We may now identify B with $C_\infty(E)$ and observe that the positive linear functional μ obtained in §3 is a positive Radon measure on E in the sense of Bourbaki [**4**]. By (4), $B=C_\infty(E)$ is dense in V under the p-topology. But the ring of continuous functions with compact supports on E is uniformly dense in $C_\infty(E)$ and it is easily seen that the former has V as its completion under the p-topology. Thus V is linearly, latticially and isometrically isomorphic to $\mathfrak{L}^1(E, \mu)$. Clearly

$V/p^{-1}(0)$ is abstractly identical with $L^1(E, \mu)$. This completes the proof of the main theorem.

5. **The compact case.** We now consider vector lattices V satisfying (1), (2) and (3) in the definition of $(\mathfrak{A}\mathfrak{L})$-space and the additional conditions:

(4′) V is boundedly σ-complete, i.e., each countable set in V which is bounded above has a least upper bound.

(4″) There is a total family Λ of linear lattice functionals such that the subspace $B=\{x\in V: \|x\|=\sup_{\lambda\in\Lambda}|\lambda(x)|<\infty\}$ is dense in V under the p-topology and contains an element e satisfying $\lambda(e)=1$, for each $\lambda\in\Lambda$.

THEOREM 3. *Let V be a vector lattice satisfying conditions* (1)–(3), (4′) *and* (4″). *Then there is a compact Hausdorff space S and a unique positive Radon measure μ on S satisfying $\mu(e)=1$ such that V is linearly, latticially and isometrically isomorphic to $\mathfrak{L}^1(S, \mu)$. Moreover, $V/p^{-1}(0)$ is abstractly identical with $L^1(S, \mu)$.*

The proof proceeds much like the proof of the main theorem, but some additional facts are needed. The element e in (4″) will of course play the role of the constant function 1. As in §3, we obtain a unique positive linear functional μ on V satisfying $p(x)=\mu(|x|)$. If we set $\mu_1(x)=(1/\mu(e))\cdot\mu(x)$, then μ_1 is again a positive linear functional "normalized" by the condition $\mu_1(e)=1$. The semi-norm $p_1(x)=\mu_1(|x|)$ associated with μ_1 is evidently equivalent to the original semi-norm p.

As in §2, we regard B as a function lattice.

LEMMA 4. *Let V be a boundedly complete* (*resp. σ-complete*) *vector lattice, B a lattice ideal in V. Then B is boundedly complete* (*resp. σ-complete*).

PROOF. By Bourbaki [**4**, p. 21, Proposition 1], it suffices to show that any set H of positive elements, directed by $\leqq$ and bounded above has a least upper bound. By assumption, sup H exists in V and $0\leqq\sup H\leqq b$, where $b\geqq H$ is the element bounding H. Thus $\sup H\in B$ by the lattice ideal property. For the σ-complete case take H countable. Q.E.D.

Thus B is a boundedly σ-complete algebra of bounded functions containing the order unit e and normed by $\|f\|=\inf\{\alpha: |f|\leqq\alpha e\}$.

LEMMA 5. *B is complete in the uniform norm.*

PROOF. Let $\{f_i\}$ be a Cauchy sequence in B. Then given $\epsilon>0$ there exists an integer k such that $\|f_n-f_m\|<\alpha$, for m, $n\geqq k$ and $\alpha=\epsilon/2$.

Now $|f_n - f_k| \leqq \alpha e$ so $-\alpha e \leqq f_n - f_k \leqq \alpha e$ and $f_k - \alpha e \leqq f_n \leqq f_k + \alpha e$. Hence f_n is bounded above and below in B. Setting $f = \lim \sup_{n \to \infty} f_n$ we have $f \in B$ by Lemmas 1 and 4 and condition (4′); moreover $f_k - \alpha e \leqq f \leqq f_k + \alpha e$ so $|f - f_k| \leqq \alpha e$ and $\|f - f_k\| \leqq \alpha = \epsilon/2 < \epsilon$. Thus $\|f - f_i\| \to 0$ as $i \to \infty$. Q.E.D.

By Theorem 2, B is linearly, latticially and isometrically isomorphic to $C(S)$ for some compact Hausdorff space S in such a way that e corresponds to the constant function 1 on S. V is then abstractly identical with $\mathfrak{L}^1(S, \mu_1)$ and $V/p_1^{-1}(0)$ can be identified with $L^1(S, \mu_1)$. This completes the proof of Theorem 3.

The example cited in §1 seems to suggest that the locally compact Hausdorff space E obtained via the representation is closely related to Segal's "perfection" of a localizable measure space (see [6]).

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