LEBESGUE SPACES OF SUMMABLE FUNCTIONS

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1. Introduction. The classical theorem of Kakutani [1] provides an elegant characterization of $L^1$ in terms of its lattice order and special properties of the norm. Much of the difficulty in representing an abstract $(L)$-space apparently stems from the fact that the end result must in general be a vector lattice whose elements are Lebesgue classes of functions rather than functions themselves. In this paper we give a characterization of $L^1$, the space of summable functions. Toward this end, we abstract the essential properties of $L^1$ in the following:

**Definition.** An $(\mathcal{E}, L)$-space is a vector lattice $V$ with a seminorm $\rho$ satisfying:

1. $\rho(x+y) = \rho(x) + \rho(y)$ for $x, y \geq 0$.
2. $\rho(|x|) = \rho(x)$ for all $x \in V$.
3. $V$ is complete in the $\rho$-topology.
4. There is a total family $\Lambda$ of linear lattice functionals (see §2 for definition) such that the subspace

$$B = \{ x \in V : \| x \| = \sup_{\lambda \in \Lambda} | \lambda(x) | < \infty \}$$

is dense in $V$ under the $\rho$-topology, and complete under the norm topology given by $\| x \|$.

**Main Theorem.** Let $V$ be an $(\mathcal{E}, L)$-space. Then there is a locally compact Hausdorff space $E$ and a unique positive Radon measure $\mu$ on $E$ such that $V$ is linearly, latticially and isometrically isomorphic to $L^1(E, \mu)$, the space of all summable functions on $E$. Of course $V/\rho^{-1}(0)$ is then abstractly identical with $L^1(E, \mu)$.

For example, if we take $V$ to be the space $L^1(X, m)$ of summable functions on a (nontopological) finite measure space $(X, S, m)$ then conditions (1)–(3) are clearly satisfied. For $\Lambda$ we may take the “point measures” (linear lattice functionals) $\epsilon_a$, $a \in X$ where $\epsilon_a(f) = f(a)$, for $f$ in $L^1$. $B$ is then the Banach algebra of bounded summable functions and (4) is clearly satisfied.

If we strengthen the above conditions by requiring $\rho$ to be a norm so that $V$ becomes an $(AL)$-space in Kakutani's sense, then it appears likely that the only such spaces are the $l^1$ spaces of summable func-

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773
tions over a discrete space relative to an atomic measure. This question will be considered in a later paper.

2. The auxiliary representation. By a linear lattice functional we mean a linear functional \( \lambda : V \to R \) satisfying \( \lambda(x^+) = (\lambda x)^+ \), for all \( x \in V \), where \( x^+ = x \vee 0 \). Such a functional is evidently positive. Regarding the existence of such functionals we have the following result (Nakayama [3]):

**Theorem 1.** Let \( V \) be a vector lattice. Then there exists a set \( X \) such that \( V \) is isomorphic, as a vector lattice, to a pointwise linear sublattice of \( R^X \) if and only if \( V \) has a total family of linear lattice functionals.

Thus condition (4) amounts, in part, to assuming that \( V \) is a function lattice. Regarding (4) we observe further that the set \( B \) inherits some rather special properties. For a set \( \Lambda \), we denote by \( m(\Lambda) \) the algebra of all bounded real-valued functions on \( \Lambda \).

**Lemma 1.** Let \( V \) be a pointwise vector lattice of functions on a set \( \Lambda \) and let \( B = m(\Lambda) \cap V \). Then \( B \) is a lattice ideal in \( V \), and, a fortiori, an algebra.

**Proof.** Let \( f \in m(\Lambda) \) with \( ||f|| > 1 \) where \( ||f|| = \sup \{ |f(\lambda)| : \lambda \in \Lambda \} \). Then \( ||f^+|| \leq ||f^2|| \); for if we take \( \lambda \in \Lambda \) so that \( |f(\lambda)| > 1 \), then \( |f(\lambda)| < |f^2(\lambda)| \). Again, if \( f \in m(\Lambda), ||f|| > 1 \), then \( f^2 \leq \|f^2\| \cdot |f| \); for we have \( \|f^2\| \cdot |f| - |f^2| = (\|f^2\| - |f|) \cdot |f| \geq 0 \) since \( |f| \leq ||f|| \leq ||f^2|| \). Now if \( g \in B \) and \( f \in V \) with \( |f| \leq |g| \), then \( ||f|| \leq ||g|| < \infty \), so \( f \in B \), i.e., \( B \) is a lattice ideal in \( V \). For \( 0 \neq f \in B \) we choose \( 0 < \epsilon < ||f|| \). Then \( g = (1/\epsilon)f \) has norm \( > 1 \). Hence \( 0 \leq g^2 \leq ||g^2|| \cdot |f| = (||g^2||/\epsilon) \cdot |f| \) and by the lattice ideal property \( g^2 \in B \). But \( f^2 = \epsilon g^2 \), so \( f \in B \). Q.E.D.

Reverting now to Theorem 1, we map \( V \) into \( R^\Lambda \) by \( x \to \hat{x} \) where \( \hat{x}(\lambda) = \lambda(x) \) for \( x \in V, \lambda \in \Lambda \). The elements of \( V \) then appear as functions on \( \Lambda \); consequently we shall identify \( V \) and \( B \) with their images under this representation and define the uniform norm on \( B \) by:

\[ ||f|| = \sup \{ |f(\lambda)| : \lambda \in \Lambda \} = \sup \{ |\lambda(f)| : \lambda \in \Lambda \} \text{ for } f \in B. \]

Condition (4) assures us that \( ||f|| \) is finite and that \( B \) is sufficiently large. It should be remarked that this representation is only a vehicle and will be discarded as soon as the desired function space appears.

3. The integral. We now focus our attention on conditions (1) and (2). Let \( V^+ = \{ x \in V : x \geq 0 \} \).

**Lemma 2.** Let \( v : V^+ \to R \) be a positive additive functional. Then there is a unique positive linear functional \( \mu : V \to R \) agreeing with \( v \) on \( V^+ \).

**Proof.** See Bourbaki [4, p. 34, Proposition 2].
Lemma 3. Let $V$ be a vector lattice with a semi-norm $p$. Then (1) and (2) hold if and only if there is a positive linear functional $\mu$ on $V$ such that $p(x) = \mu(|x|)$, for all $x \in V$. Moreover, if (1) and (2) hold, $\mu$ is uniquely determined.

**Proof.** Sufficiency. In any vector lattice we have the identities:

\[(1') |x + y| = |x| + |y| \quad \text{for } x, y \geq 0.\]
\[(2') \left( \frac{|x|}{|y|} \right) = \frac{|x|}{|y|} \quad \text{for all } x \in V.\]

The additivity of $\mu$ together with the assumption $p(x) = \mu(|x|)$ easily imply (1) and (2).

Necessity and uniqueness. Condition (1) says that $p$ is additive on the positive cone $V^+$. As in the proof of Lemma 2, we define $\mu(x) = p(x^+) - p(x^-)$. Clearly $p(x) = \mu(|x|) = \mu(|x|)$. Q.E.D.

4. Representation of $B$. The fact that $B$ is a pointwise algebra of bounded functions (Lemma 1) enables us to adjoin an order unit $e$ (the constant function 1) to $B$ (for definition of an order unit see Kadison [5, p. 3]). Thus for any function $f$ in $B$ we can find an $\alpha \geq 0$ such that $|f| \leq \alpha e$. Let $B_\alpha$ denote the algebra resulting from the adjunction of $e$ to $B$. Being a function lattice, $B_\alpha$ is manifestly Archimedean in Kadison's sense and the "natural norm" $\|f\| = \inf \{ \alpha : |f| \leq \alpha e \}$ is just the uniform norm inherited from $m(\Lambda)$. By condition (4), $B$ is complete in the uniform norm as is $B_\alpha$.

We now appeal to a classical representation theorem for vector lattices.

**Theorem 2.** Let $L$ be an Archimedean vector lattice with order unit $e$. If $L$ is complete in the norm $\|f\| = \inf \{ \alpha : |f| \leq \alpha e \}$ then one can find a compact Hausdorff space $S$ and a linear lattice isometry of $L$ with $C(S)$.

For the proof, we refer to [2, p. 103, Theorem 3] or to [5, p. 10, Theorem 4.1 and Historical Remarks].

**Corollary.** $B$ is linearly, latticially and isometrically isomorphic to $C_\alpha(E)$, the Banach algebra of all continuous functions on $E = S - \{ \infty \}$ (locally compact Hausdorff) vanishing at $\infty$.

We may now identify $B$ with $C_\alpha(E)$ and observe that the positive linear functional $\mu$ obtained in §3 is a positive Radon measure on $E$ in the sense of Bourbaki [4]. By (4), $B = C_\alpha(E)$ is dense in $V$ under the $p$-topology. But the ring of continuous functions with compact supports on $E$ is uniformly dense in $C_\alpha(E)$ and it is easily seen that the former has $V$ as its completion under the $p$-topology. Thus $V$ is linearly, latticially and isometrically isomorphic to $\mathcal{E}^1(E, \mu)$. Clearly
$V/p^{-1}(0)$ is abstractly identical with $L^1(E, \mu)$. This completes the proof of the main theorem.

5. The compact case. We now consider vector lattices $V$ satisfying (1), (2) and (3) in the definition of $(\mathfrak{E}, \mathfrak{L})$-space and the additional conditions:

$(4')$ $V$ is boundedly $\sigma$-complete, i.e., each countable set in $V$ which is bounded above has a least upper bound.

$(4'')$ There is a total family $\Lambda$ of linear lattice functionals such that the subspace $B = \{x \in V : \|x\| = \sup_{\lambda \in \Lambda} |\lambda(x)| \leq \infty \}$ is dense in $V$ under the $p$-topology and contains an element $e$ satisfying $\lambda(e) = 1$, for each $\lambda \in \Lambda$.

**Theorem 3.** Let $V$ be a vector lattice satisfying conditions (1)-(3), (4') and (4''). Then there is a compact Hausdorff space $S$ and a unique positive Radon measure $\mu$ on $S$ satisfying $\mu(e) = 1$ such that $V$ is linearly, latticially and isometrically isomorphic to $L^1(S, \mu)$. Moreover, $V/p^{-1}(0)$ is abstractly identical with $L^1(S, \mu)$.

The proof proceeds much like the proof of the main theorem, but some additional facts are needed. The element $e$ in $(4'')$ will of course play the role of the constant function 1. As in §3, we obtain a unique positive linear functional $\mu$ on $V$ satisfying $\mu(x) = \mu(|x|)$. If we set $\mu_1(x) = (1/\mu(e)) \cdot \mu(x)$, then $\mu_1$ is again a positive linear functional “normalized” by the condition $\mu_1(e) = 1$. The semi-norm $p_1(x) = \mu_1(|x|)$ associated with $\mu_1$ is evidently equivalent to the original semi-norm $p$.

As in §2, we regard $B$ as a function lattice.

**Lemma 4.** Let $V$ be a boundedly complete (resp. $\sigma$-complete) vector lattice, $B$ a lattice ideal in $V$. Then $B$ is boundedly complete (resp. $\sigma$-complete).

**Proof.** By Bourbaki [4, p. 21, Proposition 1], it suffices to show that any set $H$ of positive elements, directed by $\leq$ and bounded above has a least upper bound. By assumption, $\operatorname{sup} H$ exists in $V$ and $0 \leq \operatorname{sup} H \leq b$, where $b \geq H$ is the element bounding $H$. Thus $\operatorname{sup} H \subseteq B$ by the lattice ideal property. For the $\sigma$-complete case take $H$ countable. Q.E.D.

Thus $B$ is a boundedly $\sigma$-complete algebra of bounded functions containing the order unit $e$ and normed by $\|f\| = \inf \{\alpha : |f| \leq \alpha e\}$.

**Lemma 5.** $B$ is complete in the uniform norm.

**Proof.** Let $\{f_n\}$ be a Cauchy sequence in $B$. Then given $\varepsilon > 0$ there exists an integer $k$ such that $\|f_n - f_m\| < \alpha$, for $m, n \geq k$ and $\alpha = \varepsilon/2$. 

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Now $|f_n - f_k| \leq \alpha \varepsilon$ so $-\alpha \varepsilon \leq f_n - f_k \leq \alpha \varepsilon$ and $f_k - \alpha \varepsilon \leq f_n \leq f_k + \alpha \varepsilon$. Hence $f_n$ is bounded above and below in $B$. Setting $f = \limsup_{n \to \infty} f_n$ we have $f \in B$ by Lemmas 1 and 4 and condition (4'); moreover $f_k - \alpha \varepsilon \leq f \leq f_k + \alpha \varepsilon$ so $|f - f_k| \leq \alpha \varepsilon$ and $||f - f_k|| \leq \alpha = \varepsilon / 2 < \varepsilon$. Thus $||f - f_i|| \to 0$ as $i \to \infty$. Q.E.D.

By Theorem 2, $B$ is linearly, latticially and isometrically isomorphic to $C(S)$ for some compact Hausdorff space $S$ in such a way that $\varepsilon$ corresponds to the constant function 1 on $S$. $V$ is then abstractly identical with $L^1(S, \mu_1)$ and $V / \mathcal{P}^{-1}(0)$ can be identified with $L^1(S, \mu_1)$. This completes the proof of Theorem 3.

The example cited in §1 seems to suggest that the locally compact Hausdorff space $E$ obtained via the representation is closely related to Segal's "perfection" of a localizable measure space (see [6]).

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