get a sequence $S_1, S_2, \cdots$, of $(n-1)$-spheres such that $[S_i, S_{i+1}]$ is an n-annulus and $X = I(S_1) \cup [S_1, S_2] \cup [S_3, S_4] \cup \cdots$. Evidently X is homeomorphic to E^n.

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THREADS WITHOUT IDEMPOTENTS

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If a thread S has no idempotents and if $S^2 = S$, then S is iseomorphic with the real interval (0, 1) under ordinary multiplication [2, Corollary 5.6]. Although the result is not nearly as pleasing as the special case just quoted, we shall give here a description of any thread without idempotents. Recall from [1] that a thread is a connected topological semigroup in which the topology is that induced by a total order.

First some examples. Let X be a totally ordered set which is a connected space in the interval topology, let T be a subset of X containing, with t, all elements less than t, and let ϕ be any continuous function from X into (0, 1) whose restriction, ϕ_0, to T is a strictly order-preserving map of T onto $(0, a^2)$ where $a = \text{l.u.b. } \phi(X)$. (We admit that a might be 1.) For such a ϕ to exist it is evidently necessary that X not have a least element, that T not have a greatest element and, provided $T \neq X$ so that the least upper bound, q, of T exists, that $\phi(q) = a^2$.

If $\phi(X)$ is the open interval $(0, a)$, define a multiplication in X by: $x \circ y = \phi_0^{-1}(\phi(x)\phi(y))$. With this definition it is quite easy to see that X is a thread without idempotents and that ϕ is a homomorphism.

In the event that $\phi(X)$ is the half closed interval $(0, a]$ (which implies of course that $a < 1$), put $A = \phi^{-1}(a)$ and $B = \phi^{-1}(a^2)$, observe that q must be the least element of B, and let ψ be any continuous

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function from $A\times A$ into B taking the boundary of $A\times A$ in $S\times S$ onto q. Now define $\circ$ on S by:

$$x \circ y = \begin{cases} \phi_0^{-1}(\phi(x)\phi(y)), & \text{if } (x, y) \notin A \times A, \\ \psi(x, y), & \text{if } (x, y) \in A \times A. \end{cases}$$

If $(x, y)\notin A\times A$, then $\phi(x)\phi(y)<a^2$ so that $\phi_0^{-1}(\phi(x)\phi(y))$ is in fact well defined. Also it is clear that $\circ$ is continuous at (x, y). Again, the continuity of $\circ$ at any point in the interior of $A\times A$ follows from that of ψ. Now suppose that (x, y) is in the boundary of $A\times A$ and that $z<x\circ y<w$. Then $x\circ y=q$ and we may choose an element b in $(0, a)$ so that $\phi(z)<b^2<a^2$. Since ψ and ϕ are continuous, there exist open sets U and V containing x and y respectively such that $\psi(U\times V\cap A\times A)\subset[q, w)$ and such that $b<\phi(U)\cup\phi(V)\leqq a$. If $x'\in U$, if $y'\in V$ and if $(x', y')\in A\times A$ then $x'\circ y'=\psi(x', y')\in[q, w)$, while if $(x', y')\notin A\times A$ then $x'\circ y'=\phi_0^{-1}(\phi(x')\phi(y'))\subset\phi_0^{-1}((b^2, a^2))\subset(z, q)$. Hence $\circ$ is continuous.

Since ϕ is clearly a homomorphism, $\phi(X\circ X)=\phi(X)\phi(X)\leqq a^2<a$ and thus A does not meet $X\circ X$. Thus, for any x, y and z, $(x\circ y)\circ z$ $=\phi_0^{-1}(\phi(x\circ y)\phi(z))=\phi_0^{-1}(\phi(x)\phi(y)\phi(z))=x\circ(y\circ z)$. Again we have shown that X is a thread, and obviously no element of X is an idempotent.

Now let S be a thread which contains no idempotents. Since S is iseomorphic with its order dual, we may as well assume that $z^2<z$ for some z in S. Then $xy<\min\{x, y\}$ for each x and y in S [**2**, Lemma 3.1], and from this it is clear that a zero could be adjoined to S as a minimal element—the result again being a thread. Without actually adjoining a zero, we shall write $(0, t)=\{x \mid x<t\}$. Observe that, since the continuous image of a connected space is connected, $(0, xy]\subset(0, x]y$ for any x and y. In particular, Sy contains $(0, t]$ whenever it contains t, so that each Sy meets each Sx.

Define a relation ρ in S by: $x\rho y$ if and only if there exists a t such that $tx=ty$. If $tx=ty$ and $sy=sz$ then, choosing u and v so that $ut=vs$, $(ut)x=(ut)z$. Again, choosing u and v so that $ut=vw$, $v(wx)=v(wy)$. Since $t(xw)=t(yw)$ is obvious, ρ is a congruence relation on S. Letting θ be the natural homomorphism of S onto S/ρ, $\theta(x)$ is the congruence class containing x.

If t, x and s are any elements in S then $t(xs)<\min\{tx, s\}\leqq tx$. That is, no element in xS can be congruent to x, so each congruence class modulo ρ is bounded from below.

LEMMA 1. *If $r=\text{g.l.b. }\theta(x)$, then each congruence class meets Sr in at most one point.*

PROOF. We must show that $z\rho w$ is impossible whenever z and w are in Sr and $z<w$. Assuming first that $w<ur$ for some u in S, choose y in (z, w) and let $v=\text{l.u.b. } \{s \mid s\leqq u \text{ and } y=sr\}$. Then $vr=y<w<ur$. Furthermore, $z<[v, u]r$, because if $sr=z$ for some s in $[v, u]$ then $v<s$ and $y\in[s, u]r$ contrary to the definition of v. Consequently, there exists an open interval Q containing r such that $vQ<w<uQ$ while $z<[v, u]Q$. Since $r=\text{g.l.b. } \theta(x)$, there exists a w' in Q such that $w'\rho x$. Now $vw'<w<uw'$ implies that $w=kw'$ for some k in $[v, u]$, and $z<kw'$ implies that $z=kz'$ for some z' less than w'. Since $k\in[v, u]$, z' cannot be in Q and thus $z'<r$. This means that z' and w' are not congruent, and since $z=kz'$ and $w=kw'$, neither are z and w.

We have shown that $z\rho w$ is impossible when w is not a maximal element in Sr. Now suppose that w is any element in Sr and that $tz=tw$. Choosing y in (z, w), $tz=ty$ is impossible because y is not maximal in Sr. But if $ty<tz$, then $ty\in t(0, z]$ so that $ty=tz'$ where z' is an element less than y. While if $tz<ty$ and if $s\in(tz, ty)=(tw, ty)$, then $s=tz'=tw'$ where $z<z'<y<w'<w$. Hence, each congruence class meets Sr in at most one point.

COROLLARY 1. *$z\rho w$ if and only if $zt=wt$ for some t in S; S/ρ is a cancellative semigroup.*

PROOF. Define another congruence on S by: $x\sigma y$ if and only if there exists a t such that $xt=yt$. Let $r=\text{g.l.b. } \theta(x)$ and let $z\rho w$. Then $zr\rho wr$ while both zr and wr belong to Sr, so by the lemma, $zr=wr$. Thus $z\rho w$ implies $z\sigma w$, and by symmetry $\rho=\sigma$. That S/ρ is a cancellative semigroup is now a familiar fact.

COROLLARY 2. *If $r=\text{g.l.b. } \theta(x)$, then $z\rho w$ if and only if $rz=rw$; each congruence class is closed.*

PROOF. It was shown in the proof of Corollary 1 that $z\rho w$ implies $zr=wr$, and by symmetry, $z\sigma w$ implies $rz=rw$. Thus $z\rho w$ if and only if $rz=rw$, and from this it follows immediately that each class is closed.

Now fix a p which is the least element of its congruence class, and define an order on S/ρ by: $\theta(x)<\theta(y)$ if and only if $px<py$. This certainly defines a total order on S/ρ because, by Corollary 2, left multiplication by p completely determines ρ. Moreover, it is easy to verify that for each x and y with $\theta(x)<\theta(y)$ there is a z such that $\theta(x)<\theta(z)<\theta(y)$, and that every subset of S/ρ which is bounded from below has a greatest lower bound. These two facts are equivalent to the assertion that S/ρ is connected in the interval topology.

To show that the multiplication in S/ρ is continuous, let W be an open set containing $\theta(x)\theta(y)$ and assume that neither $\theta(x)$ nor $\theta(y)$ is maximal in S/ρ; the other cases are quite similar. Let $\theta(x)<\theta(u)$ and observe that, since $pxs<px$ for each s, $\theta(xS)<\theta(x)$ and $xS<u$. Now put $s=\text{g.l.b.}\ \{t \mid \theta(x)<\theta(t)\}$ and $r=\text{l.u.b.}\ \{t \mid t\leqq u \text{ and } \theta(t)<\theta(x)\}$, and choose z and w with respect to y analogously. Since θ is clearly continuous, $\theta(\{r, s\}\{z, w\})=\theta(x)\theta(y)\in W$ and there exist open sets U, containing r and s, and V, containing z and w, such that $\theta(UV)\subset W$. By the definitions of r and s there is an r' less than r and an s' greater than s such that $\theta(r')<\theta(x)<\theta(s')$ and such that $[r', r]\cup[s, s']\subset U$. Since θ is continuous and $\theta(r)=\theta(s)$, $\theta([r', r])\cup\theta([s, s'])$ is a connected subset of $\theta(U)$ containing $\theta(r')$ and $\theta(s')$. Hence $\theta(U)$ contains an open interval about $\theta(x)$, and similarly, $\theta(V)$ contains an open interval about $\theta(y)$. Since $\theta(U)\theta(V)=\theta(UV)\subset W$, the multiplication in S/ρ is continuous at $(\theta(x), \theta(y))$.

LEMMA 2. *S/ρ is iseomorphic with a subthread of the real interval* $(0, 1)$ *under ordinary multiplication.*

PROOF. We have just proved that S/ρ is a cancellative thread. Since it contains no idempotents, the lemma follows from the work of Aczél and Tamari [**1**, p. 81].

THEOREM. *If S is a thread which has no idempotents, then S is iseomorphic with one of the examples given above.*

PROOF. We have shown that there exists a continuous homomorphism ϕ from S into $(0, 1)$ such that ρ is the congruence relation determined in S by ϕ. That is, $\phi(x)=\phi(y)$ if and only if $px=py$. Let $a=\text{l.u.b.}\ \phi(S)$, let $A=\phi^{-1}(a)$ and let $B=\phi^{-1}(a^2)$. Of course, A may be empty; in fact, A and B are both empty if $a=1$. If B is not empty let q be its least element and let $T=\{t \mid t<q\}$; otherwise put $T=S$.

Now $xy\notin T$ implies that x and y are both in A. Indeed, if $xy\notin T$ then $q\leqq xy$ and $q=xy'$ for some y' in S. Then $a^2=\phi(q)=\phi(x)\phi(y')\leqq\phi(x)a$ implies that $\phi(x)=a$. Likewise, $q\in Sy$ so that $\phi(y)=a$.

If z is the least element of some congruence class modulo ρ, then Sz meets each class in at most one point. That is, ϕ is one-to-one on Sz. Now if $u<v$ and $\phi(v)<\phi(u)$ with u and v in Sz, then $pv<pu$. But this implies that $pv=pv'$ for some v' in $(0, u]$ and thus less than v. Consequently, ϕ is strictly order-preserving on Sz. If w is any element of T then $(0, wz]\subset Sz$ and $\phi([wz, w])$ is compact, so that $\phi((0, w])$ has a greatest element. Thus we may choose a b in $(0, a)$ such that $\phi((0, w])<b^2$. Letting x be the least element in $\phi^{-1}(b)$, it follows that $w<x^2$. But x is the least element of its congruence class, so ϕ is strictly order-preserving on Sx which contains $(0, w]$. Since such an x can be found for each w in T, ϕ is strictly order-preserving on T.

If $c < a^2$ then $c = d^2$ for some d less than a. Choosing x so that $\phi(x) = d$, $\phi(x^2) = c$ and, since x is not in A, $x^2 \in T$. Conversely, it follows from the minimality of q that $\phi(T) \subset (0, a^2)$. Thus if ϕ_0 is the restriction of ϕ to T, then ϕ_0 is a strictly order-preserving map of T onto $(0, a^2)$.

If either x or y is not in A, then $xy \in T$ and $xy = \phi_0^{-1}\phi(xy) = \phi_0^{-1}(\phi(x)\phi(y))$. If A is empty we are through.

If A is not empty, define $\psi: A \times A \rightarrow B$ by $\psi(x, y) = xy$. This is obviously a continuous function. Moreover, if $(x, y) \in A \times A$ and if $xy \neq q$, then $xy \in B$ and $q < xy$. By continuity there exist open sets U and V containing x and y such that $q < UV$. Since UV does not meet T, $U \cup V \subset A$ and (x, y) is in the interior of $A \times A$. In other words, ψ maps the boundary of $A \times A$ onto q. This completes the proof.

Since this result evidently gives us also a description of all threads which have a zero as an endpoint and which have no zero divisors, one might expect that a similar result holds even if zero divisors are present. Perhaps all threads with a zero as a least element which have zero divisors can be obtained by altering the examples above only to the extent of replacing the thread (0, 1) by the Rees quotient of [0, 1) by the ideal [0, 1/2]. We have found no other examples, but neither have we found even the beginning of a proof.

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