get a sequence $S_1, S_2, \ldots$, of $(n-1)$-spheres such that $[S_i, S_{i+1}]$ is an $n$-annulus and $X = \bigcup [S_i, S_{i+1}]$ is homeomorphic to $E^n$.

**References**


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**THREADS WITHOUT IDEMPOTENTS**

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If a thread $S$ has no idempotents and if $S^2 = S$, then $S$ is isomorphic with the real interval $(0, 1)$ under ordinary multiplication [2, Corollary 5.6]. Although the result is not nearly as pleasing as the special case just quoted, we shall give here a description of any thread without idempotents. Recall from [1] that a thread is a connected topological semigroup in which the topology is that induced by a total order.

First some examples. Let $X$ be a totally ordered set which is a connected space in the interval topology, let $T$ be a subset of $X$ containing, with $t$, all elements less than $t$, and let $\phi$ be any continuous function from $X$ into $(0, 1)$ whose restriction, $\phi_0$, to $T$ is a strictly order-preserving map of $T$ onto $(0, a^2)$ where $a = \text{l.u.b. } \phi(X)$. (We admit that $a$ might be 1.) For such a $\phi$ to exist it is evidently necessary that $X$ not have a least element, that $T$ not have a greatest element and, provided $T = X$ so that the least upper bound, $q$, of $T$ exists, that $\phi(q) = a^2$.

If $\phi(X)$ is the open interval $(0, a)$, define a multiplication in $X$ by: $x \circ y = \phi^{-1}(\phi(x)\phi(y))$. With this definition it is quite easy to see that $X$ is a thread without idempotents and that $\phi$ is a homomorphism.

In the event that $\phi(X)$ is the half closed interval $(0, a]$ (which implies of course that $a < 1$), put $A = \phi^{-1}(a)$ and $B = \phi^{-1}(a^2)$, observe that $q$ must be the least element of $B$, and let $\psi$ be any continuous

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function from \( A \times A \) into \( B \) taking the boundary of \( A \times A \) in \( S \times S \) onto \( q \). Now define \( o \) on \( S \) by:

\[
x \circ y = \begin{cases} 
\phi^{-1}(\phi(x)\phi(y)), & \text{if } (x, y) \in A \times A, \\
\psi(x, y), & \text{if } (x, y) \in A \times A.
\end{cases}
\]

If \((x, y)\in A \times A\), then \(\phi(x)\phi(y) < a^2\) so that \(\phi^{-1}(\phi(x)\phi(y))\) is in fact well defined. Also it is clear that \(o\) is continuous at \((x, y)\). Again, the continuity of \(o\) at any point in the interior of \( A \times A \) follows from that of \(\psi\). Now suppose that \((x, y)\) is in the boundary of \( A \times A \) and that \(z < x \circ y < w\). Then \(x \circ y = q\) and we may choose an element \(b\) in \((0, a)\) so that \(\phi(z) < b^2 < a^2\). Since \(\psi\) and \(\phi\) are continuous, there exist open sets \(U\) and \(V\) containing \(x\) and \(y\) respectively such that \(\psi(U \times V \cap A \times A) \subseteq [q, w]\) and such that \(b < \phi(U) \cup \phi(V) \leq a\). If \(x' \in U\), if \(y' \in V\) and if \((x', y') \in A \times A\) then \(x' \circ y' = \psi(x', y') \subseteq [q, w]\), while if \((x', y') \in A \times A\) then \(x' \circ y' = \phi^{-1}(\phi(x')\phi(y')) \subseteq \phi^{-1}([b^2, a^2]) \subseteq (z, q)\). Hence \(o\) is continuous.

Since \(\phi\) is clearly a homomorphism, \(\phi(X \circ X) = \phi(X)\phi(X) \leq a^2 < a\) and thus \(A\) does not meet \(X \circ X\). Thus, for any \(x, y\) and \(z\), \((x \circ y) \circ z = \phi^{-1}(\phi(x \circ y)\phi(z)) = \phi^{-1}(\phi(x)\phi(y)\phi(z)) = x \circ (y \circ z)\). Again we have shown that \(X\) is a thread, and obviously no element of \(X\) is an idempotent.

Now let \(S\) be a thread which contains no idempotents. Since \(S\) is isomorphic with its order dual, we may as well assume that \(z^2 < z\) for some \(z\) in \(S\). Then \(xy < \min\{x, y\}\) for each \(x\) and \(y\) in \(S\) [2, Lemma 3.1], and from this it is clear that a zero could be adjoined to \(S\) as a minimal element, the result again being a thread. Without actually adjoining a zero, we shall write \((0, t) = \{x|x < t\}\). Observe that, since the continuous image of a connected space is connected, \((0, xy) \subseteq (0, x) y\) for any \(x\) and \(y\). In particular, \(Sy\) contains \((0, t)\) whenever it contains \(t\), so that each \(Sy\) meets each \(Sx\).

Define a relation \(\rho\) in \(S\) by: \(x\rho y\) if and only if there exists a \(t\) such that \(tx = ty\). If \(tx = ty\) and \(sy = sz\) then, choosing \(u\) and \(v\) so that \(ut = vs\), \((ut)x = (ut)z\). Again, choosing \(u\) and \(v\) so that \(ut = vw\), \(v(wx) = v(wy)\). Since \(t(xw) = t(yw)\) is obvious, \(\rho\) is a congruence relation on \(S\). Letting \(\theta\) be the natural homomorphism of \(S\) onto \(S/\rho\), \(\theta(x)\) is the congruence class containing \(x\).

If \(t, x\) and \(s\) are any elements in \(S\) then \(t(xs) < \min\{tx, s\} \leq tx\). That is, no element in \(xS\) can be congruent to \(x\), so each congruence class modulo \(\rho\) is bounded from below.

**Lemma 1.** If \(r = \text{g.l.b.} \theta(x)\), then each congruence class meets \(Sr\) in at most one point.
Proof. We must show that $zp \leq w$ is impossible whenever $z$ and $w$ are in $Sr$ and $z < w$. Assuming first that $w < ur$ for some $u$ in $S$, choose $y$ in $(z, w)$ and let $v = l.u.b. \{ s \mid s \leq u$ and $y = sr \}$. Then $vr = y < w < ur$. Furthermore, $z < [v, u]r$, because if $sr = z$ for some $s$ in $[v, u]$ then $v < s$ and $y \subseteq [s, u]r$ contrary to the definition of $v$. Consequently, there exists an open interval $Q$ containing $r$ such that $vQ < w < uQ$ while $z < [v, u]Q$. Since $r = g.l.b. \theta(x)$, there exists a $w'$ in $Q$ such that $w' \leq x$. Now $vw' < w < uw'$ implies that $w = kw'$ for some $k$ in $[v, u]$, and $z < kw'$ implies that $z = kz'$ for some $z'$ less than $w'$. Since $k \in [v, u]$, $z'$ cannot be in $Q$ and thus $z' < x$. This means that $z'$ and $w'$ are not congruent, and since $z = kz'$ and $w = kw'$, neither are $z$ and $w$. We have shown that $zp \leq w$ is impossible when $w$ is not a maximal element in $Sr$. Now suppose that $w$ is any element in $Sr$ and that $tz = tw$. Choosing $y$ in $(z, w)$, $tz = ty$ is impossible because $y$ is not maximal in $Sr$. But if $ty < tz$, then $ty \subseteq t(0, z]$ so that $ty = ts'$ where $s'$ is an element less than $y$. While if $ts < ty$ and if $s \subseteq (ts, ty) = (tw, ty)$, then $s = ts' = tw'$ where $z < z' < y < w' < w$. Hence, each congruence class meets $Sr$ in at most one point.

Corollary 1. $zp \leq w$ if and only if $zt = wt$ for some $t$ in $S$; $S/\rho$ is a cancellative semigroup.

Proof. Define another congruence on $S$ by: $xc \equiv y$ if and only if there exists a $t$ such that $xt = yt$. Let $r = g.l.b. \theta(x)$ and let $zp \leq w$. Then $zr = wr$ while both $zr$ and $wr$ belong to $Sr$, so by the lemma, $zr = wr$. Thus $zp \leq w$ implies $zr = wr$, and by symmetry $\rho = \sigma$. That $S/\rho$ is a cancellative semigroup is now a familiar fact.

Corollary 2. If $r = g.l.b. \theta(x)$, then $zp \leq w$ if and only if $rz = rw$; each congruence class is closed.

Proof. It was shown in the proof of Corollary 1 that $zp \leq w$ implies $zr = wr$, and by symmetry, $zr \leq w$ implies $rz = rw$. Thus $zp \leq w$ if and only if $rz = rw$, and from this it follows immediately that each class is closed. Now fix a $p$ which is the least element of its congruence class, and define an order on $S/\rho$ by: $\theta(x) < \theta(y)$ if and only if $px < py$. This certainly defines a total order on $S/\rho$ because, by Corollary 2, left multiplication by $p$ completely determines $\rho$. Moreover, it is easy to verify that for each $x$ and $y$ with $\theta(x) < \theta(y)$ there is a $z$ such that $\theta(x) < \theta(z) < \theta(y)$, and that every subset of $S/\rho$ which is bounded from below has a greatest lower bound. These two facts are equivalent to the assertion that $S/\rho$ is connected in the interval topology.

To show that the multiplication in $S/\rho$ is continuous, let $W$ be an open set containing $\theta(x)\theta(y)$ and assume that neither $\theta(x)$ nor $\theta(y)$ is
maximal in $S/\rho$; the other cases are quite similar. Let $\theta(x) < \theta(u)$ and observe that, since $\rho x s < \rho x$ for each $s$, $\theta(xs) < \theta(x)$ and $xS < u$. Now put $s = \text{g.l.b.} \{ t \mid \theta(x) < \theta(t) \}$ and $r = \text{l.u.b.} \{ t \mid t \leq u \text{ and } \theta(t) < \theta(x) \}$, and choose $z$ and $w$ with respect to $y$ analogously. Since $\theta$ is clearly continuous, $\theta(\{ r, s \} \{ z, w \}) = \theta(x)\theta(y) \subseteq W$ and there exist open sets $U$, containing $r$ and $s$, and $V$, containing $z$ and $w$, such that $\theta(UV) \subseteq W$.

By the definitions of $r$ and $s$ there is an $r'$ less than $r$ and an $s'$ greater than $s$ such that $\theta(r') < \theta(x) < \theta(s')$ and such that $[r', r] \cup [s, s'] \subseteq U$. Since $\theta$ is continuous and $\theta(r) = \theta(s)$, $\theta([r', r] \cup [s, s'])$ is a connected subset of $\theta(U)$ containing $\theta(r')$ and $\theta(s')$. Hence $\theta(U)$ contains an open interval about $\theta(x)$, and similarly, $\theta(V)$ contains an open interval about $\theta(y)$. Since $\theta(U)\theta(V) = \theta(UV) \subseteq W$, the multiplication in $S/\rho$ is continuous at $(\theta(x), \theta(y))$.

**Lemma 2.** $S/\rho$ is isomorphic with a subthread of the real interval $(0, 1)$ under ordinary multiplication.

**Proof.** We have just proved that $S/\rho$ is a cancellative thread. Since it contains no idempotents, the lemma follows from the work of Aczél and Tamari [1, p. 81].

**Theorem.** If $S$ is a thread which has no idempotents, then $S$ is isomorphic with one of the examples given above.

**Proof.** We have shown that there exists a continuous homomorphism $\phi$ from $S$ into $(0, 1)$ such that $\rho$ is the congruence relation determined in $S$ by $\phi$. That is, $\phi(x) = \phi(y)$ if and only if $\rho x = \rho y$. Let $a = \text{l.u.b.} \phi(S)$, let $A = \phi^{-1}(a)$ and let $B = \phi^{-1}(a^2)$. Of course, $A$ may be empty; in fact, $A$ and $B$ are both empty if $a = 1$. If $B$ is not empty let $q$ be its least element and let $T = \{ t \mid t < q \}$; otherwise put $T = S$.

Now $xy \in T$ implies that $x$ and $y$ are both in $A$. Indeed, if $xy \in T$ then $q \leq xy$ and $q = xy'$ for some $y'$ in $S$. Then $a^2 = \phi(q) = \phi(x)\phi(y') \leq \phi(x)a$ implies that $\phi(x) = a$. Likewise, $q \leq Sy$ so that $\phi(y) = a$.

If $z$ is the least element of some congruence class modulo $\rho$, then $Sz$ meets each class in at most one point. That is, $\phi$ is one-to-one on $Sz$. Now if $u < v$ and $\phi(v) < \phi(u)$ with $u$ and $v$ in $Sz$, then $\rho v < \rho u$. But this implies that $\rho v = \rho v'$ for some $v'$ in $(0, u]$ and thus less than $v$. Consequently, $\phi$ is strictly order-preserving on $Sz$. If $w$ is any element of $T$ then $(0, wx] \subseteq Sz$ and $\phi((wx, w])$ is compact, so that $\phi((0, w])$ has a greatest element. Thus we may choose a $b$ in $(0, a)$ such that $\phi((0, w]) = b^2$. Letting $x$ be the least element in $\phi^{-1}(b)$, it follows that $w < a^2$. But $x$ is the least element of its congruence class, so $\phi$ is strictly order-preserving on $Sx$ which contains $(0, w]$. Since such an $x$ can be found for each $w$ in $T$, $\phi$ is strictly order-preserving on $T$.
If \( c < a^2 \) then \( c = d^2 \) for some \( d \) less than \( a \). Choosing \( x \) so that \( \phi(x) = d \), \( \phi(x^2) = c \) and, since \( x \) is not in \( A \), \( x^2 \in T \). Conversely, it follows from the minimality of \( q \) that \( \phi(T) \subseteq (0, a^2) \). Thus if \( \phi_0 \) is the restriction of \( \phi \) to \( T \), then \( \phi_0 \) is a strictly order-preserving map of \( T \) onto \( (0, a^2) \).

If either \( x \) or \( y \) is not in \( A \), then \( xy \in T \) and \( xy = \phi_0^{-1}(\phi(xy)) = \phi_0^{-1}(\phi(x)\phi(y)) \). If \( A \) is empty we are through.

If \( A \) is not empty, define \( \psi : A \times A \to B \) by \( \psi(x, y) = xy \). This is obviously a continuous function. Moreover, if \( (x, y) \in A \times A \) and if \( xy \neq q \), then \( xy \in B \) and \( q < xy \). By continuity there exist open sets \( U \) and \( V \) containing \( x \) and \( y \) such that \( q < UV \). Since \( UV \) does not meet \( T \), \( U \cup V \subseteq A \) and \( (x, y) \) is in the interior of \( A \times A \). In other words, \( \psi \) maps the boundary of \( A \times A \) onto \( q \). This completes the proof.

Since this result evidently gives us also a description of all threads which have a zero as an endpoint and which have no zero divisors, one might expect that a similar result holds even if zero divisors are present. Perhaps all threads with a zero as a least element which have zero divisors can be obtained by altering the examples above only to the extent of replacing the thread \( (0, 1) \) by the Rees quotient of \( [0, 1) \) by the ideal \( [0, 1/2] \). We have found no other examples, but neither have we found even the beginning of a proof.

References


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