THE MONOTONE UNION OF OPEN $n$-CELLS
IS AN OPEN $n$-CELL

MORTON BROWN

In a research announcement [2] B. Mazur indicated that modulo
the Generalized Schoenflies Theorem, the following theorem could be
proved:

"If the open cone over a topological space $X$ is locally Euclidean at
the origin, then it is topologically equivalent with Euclidean space."

Ronald Rosen [3] has described an ingenious proof of this theorem
based on the now known [1] Generalized Schoenflies Theorem. In the
present paper we prove a stronger theorem without employing the
Generalized Schoenflies Theorem.

DEFINITIONS AND NOTATION. If $Q$ is an $n$-cell then $\hat{Q}, \dot{Q}$ denote the
interior and boundary of $Q$, respectively. An $n$-annulus is a homeo-
morph of $S^{n-1}\times [01)$. If $S$ is an $(n-1)$-sphere in an $n$-cell, then $I(S)$
denotes the interior (complementary domain) of $S$. If $S_1, S_2$ are
$(n-1)$-spheres in an $n$-cell and $S_1 \subseteq I(S_2)$, then $[S_1, S_2]$ (or equival-
ently $[S_2, S_1]$) denotes the set $C \{I(S_2)\} - I(S_1)$. An $(n-1)$-sphere $S$
embedded in a space $X$ is collared if there is a homeomorphism $h$ of
$S^{n-1}\times [01)$ into $X$ such that $h(S^{n-1}\times 1/2) = S$. Finally $B$, will denote
the $n$-ball of radius $r$ in $E^n$ and centered at the origin.

**Lemma 1.** Let $S$ be a collared $(n-1)$-sphere in the interior of an $n$-cell
$Q$ such that $C \{I(S)\}$ is an $n$-cell. Let $h$ be a homeomorphism of $Q$ upon
itself such that $S \subseteq I(h(S))$ and $h|U = 1$ where $U$ is a nonempty open
subset of $I(S)$. Then $h(S)$ is a collared $(n-1)$-sphere in $\hat{Q}$, $C \{I(h(S))\}$
is an $n$-cell and $[S, h(S)]$ is an $n$-annulus.

**Proof.** Let $f$ be a homeomorphism of $S^{n-1}\times [01)$ into $\hat{Q}$ such that
$f(S^{n-1}\times 0) = S, f(S^{n-1}\times [01]) \cap h(S) = 0$, and $f(S^{n-1}\times [01]) \cap I(S) = 0$.
Evidently $I(S) \cup f(S^{n-1}\times [01])$ is an $n$-cell. Hence there is a homeo-
morphism $g$ of $Q$ upon itself such that:

1. $g(S) \subseteq U$,
2. $g|f(S^{n-1}\times 0) \cap 1/2) = S$,
3. $g| h(S) = 1$.

Then

Presented to the Society, September 2, 1960; received by the editors September
30, 1960.

1 The results of [1] make this last part of the hypothesis unnecessary.
THE MONOTONE UNION OF OPEN \( n \)-CELLS

\[
g^{-1} h g_f(S^{n-1} \times [0, 1/2]) = g^{-1} h g_f(S^{n-1} \times 0, f(S^{n-1} \times 1/2))
\]
\[
= g^{-1} h [g(S), S]
\]
\[
= g^{-1} [g(S), h(S)]
\]
\[
= [S, h(S)].
\]

Hence \([S, h(S)]\) is an \( n \)-annulus. Obviously \( h(S) \) is collared and \( \text{Cl} [I(h(S))] \) is an \( n \)-cell.

**Lemma 2.** Let \( S \) be a collared \((n - 1)\)-sphere in the interior of an \( n \)-cell \( Q \) such that \( \text{Cl} [I(S)] \) is an \( n \)-cell.1 Suppose \( M \) is a compact subset of \( Q \). Then there is a collared \((n - 1)\)-sphere \( S' \) in \( Q \) such that \( I(S') \supset M \cup S \), \( \text{Cl} [I(S')] \) is an \( n \)-cell, and \([S, S']\) is an \( n \)-annulus.

**Proof.** We may suppose without loss of generality that \( Q \) is the unit ball \( B_1 \) in \( E^n \) and that \( I(S) \) contains the origin. Let \( \epsilon > 0 \) be small enough so that \( B_\epsilon \subset I(S) \) and \( M \subset I(S) \). Let \( h \) be a homeomorphism of \( B_1 \) upon itself such that \( h|B_{1/2} = 1 \) and \( h(B_\epsilon) \supset B_{1-\epsilon} \). Then \( S' = h(S) \) contains \( M \cup S \) in its interior. Lemma 1 insures that \( h(S) \) is collared and that \([S, h(S)]\) is an \( n \)-annulus.

**Theorem.** Let \( X \) be a topological space which is the union of a sequence \( V_1 \subset V_2 \subset \cdots \subset V_i \subset \cdots \) of open subsets where each \( V_i \) is homeomorphic to \( E^n \). Then \( X \) is homeomorphic to \( E^n \).

**Proof.** Let \( h_i \) map \( E^n \) homeomorphically onto \( V_i \). Then \( h_i(B_1) \) is an \( n \)-cell in \( V_i \). There is an integer \( n_2 \) such that

\[
h_2(B_{n_2}) \supset h_1(B_2) \cup h_2(B_2).
\]

Inductively, there is a sequence of integers \( n_3, n_4, \ldots \), such that for all \( i \)

\[
h_i(B_{n_i}) \supset h_1(B_i) \cup \cdots \cup h_i(B_i) \cup h_{i-1}(B_{n-1}).
\]

Since \( X \) is locally Euclidean, \( h_i(B_{n_i}) \) is an \( n \)-cell in \( X \) containing \( h_{i-1}(B_{n_{i-1}}) \) in its interior \( h_i(B_{n_i}) \). Finally \( \bigcup_{i=1}^\infty B_{n_i} = X \). For if \( x \in X \) there is an integer \( j \) such that \( x \in V_j \). Hence there is an integer \( k > j \) such that \( x \in h_j(B_{n_k}) \). But then \( x \in h_k(B_{n_k}) \). Let \( Q_i = h_i(B_{n_k}) \). Then \( X = \bigcup_{i=1}^\infty Q_i \) where \( Q_i \) is an \( n \)-cell, \( Q_i \subset \hat{Q}_{i+1} \), and \( \hat{Q}_{i+1} \) is open in \( X \).

Let \( S_1 \) be a collared \((n - 1)\)-sphere in \( \hat{Q}_1 \) such that \( \text{Cl} [I(S_1)] \) is an \( n \)-cell. Applying Lemma 2 to the \( n \)-cell \( Q_2 \), we obtain a collared \((n - 1)\)-sphere \( S_2 \) in \( \hat{Q}_2 \) such that \( I(S_2) \supset Q_2 \cup S_1 \), \([S_1, S_2]\) is an \( n \)-annulus, and \( \text{Cl} [I(S_2)] \) is an \( n \)-cell. The same lemma applied to \( Q_3 \) and \( S_2 \) yields us a collared sphere \( S_3 \) in \( \hat{Q}_3 \) such that \( I(S_3) \supset S_3 \cup Q_2 \), \([S_2, S_3]\) is an \( n \)-annulus, and \( \text{Cl} [I(S_3)] \) is an \( n \)-cell. Continuing this argument, we
get a sequence $S_1, S_2, \ldots$, of $(n-1)$-spheres such that $[S_i, S_{i+1}]$ is an $n$-annulus and $X = I(S_1) \cup [S_1, S_2] \cup [S_3, S_4] \cup \cdots$. Evidently $X$ is homeomorphic to $E^n$.

**References**


**University of Michigan**

---

**THREADS WITHOUT IDEMPOTENTS**

C. R. STOREY

If a thread $S$ has no idempotents and if $S^2 = S$, then $S$ is isomorphic with the real interval $(0, 1)$ under ordinary multiplication [2, Corollary 5.6]. Although the result is not nearly as pleasing as the special case just quoted, we shall give here a description of any thread without idempotents. Recall from [1] that a thread is a connected topological semigroup in which the topology is that induced by a total order.

First some examples. Let $X$ be a totally ordered set which is a connected space in the interval topology, let $T$ be a subset of $X$ containing, with $t$, all elements less than $t$, and let $\phi$ be any continuous function from $X$ into $(0, 1)$ whose restriction, $\phi_0$, to $T$ is a strictly order-preserving map of $T$ onto $(0, a^2)$ where $a = \text{l.u.b. } \phi(X)$. (We admit that $a$ might be 1.) For such a $\phi$ to exist it is evidently necessary that $X$ not have a least element, that $T$ not have a greatest element and, provided $T \neq X$ so that the least upper bound, $q$, of $T$ exists, that $\phi(q) = a^2$.

If $\phi(X)$ is the open interval $(0, a)$, define a multiplication in $X$ by: $x \circ y = \phi^{-1}(\phi(x)\phi(y))$. With this definition it is quite easy to see that $X$ is a thread without idempotents and that $\phi$ is a homomorphism.

In the event that $\phi(X)$ is the half closed interval $(0, a]$ (which implies of course that $a < 1$), put $A = \phi^{-1}(a)$ and $B = \phi^{-1}(a^2)$, observe that $q$ must be the least element of $B$, and let $\psi$ be any continuous

---

Received by the editors July 28, 1960.

1 This paper was prepared while the author held a National Science Foundation Postdoctoral Fellowship.