THE MONOTONE UNION OF OPEN n-CELLS IS AN OPEN n-CELL

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In a research announcement [2] B. Mazur indicated that modulo the Generalized Schoenflies Theorem, the following theorem could be proved:

"If the open cone over a topological space X is locally Euclidean at the origin, then it is topologically equivalent with Euclidean space."

Ronald Rosen [3] has described an ingenious proof of this theorem based on the now known [1] Generalized Schoenflies Theorem. In the present paper we prove a stronger theorem without employing the Generalized Schoenflies Theorem.

DEFINITIONS AND NOTATION. If Q is an n-cell then $\overset{\circ}{Q}$, $\dot{Q}$ denote the interior and boundary of Q, respectively. An *n-annulus* is a homeomorph of $S^{n-1}\times[01]$. If S is an $(n-1)$-sphere in an n-cell, then $I(S)$ denotes the interior (complementary domain) of S. If S_1, S_2 are $(n-1)$-spheres in an n-cell and $S_1\subset I(S_2)$, then $[S_1, S_2]$ (or equivalently $[S_2, S_1]$) denotes the set $\mathrm{Cl}\,[I(S_2)]-I(S_1)$. An $(n-1)$-sphere S embedded in a space X is *collared* if there is a homeomorphism h of $S^{n-1}\times[01]$ into X such that $h(S^{n-1}\times 1/2)=S$. Finally B_r will denote the n-ball of radius r in E^n and centered at the origin.

LEMMA 1. *Let S be a collared $(n-1)$-sphere in the interior of an n-cell Q such that* $\mathrm{Cl}\,[I(S)]$ *is an n-cell.*[1] *Let h be a homeomorphism of Q upon itself such that $S\subset I(h(S))$ and $h|U=1$ where U is a nonempty open subset of $I(S)$. Then $h(S)$ is a collared $(n-1)$-sphere in $\overset{\circ}{Q}$,* $\mathrm{Cl}\,[I(h(S))]$ *is an n-cell and $[S, h(S)]$ is an n-annulus.*

PROOF. Let f be a homeomorphism of $S^{n-1}\times[01]$ into $\overset{\circ}{Q}$ such that $f(S^{n-1}\times 0)=S$, $f(S^{n-1}\times[01])\cap h(S)=0$, and $f(S^{n-1}\times[01])\cap I(S)=0$. Evidently $I(S)\cup f(S^{n-1}\times[01])$ is an n-cell. Hence there is a homeomorphism g of Q upon itself such that:

$$g(S)\subset U, \tag{1}$$

$$gf(S^{n-1}\times 1/2)=S, \tag{2}$$

$$g\,|\,h(S)=1. \tag{3}$$

Then

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[1] The results of [1] make this last part of the hypothesis unnecessary.

$$\begin{aligned} g^{-1}hgf(S^{n-1}\times[0,1/2]) &= g^{-1}hg[f(S^{n-1}\times 0), f(S^{n-1}\times 1/2)] \\ &= g^{-1}h[g(S), S] \\ &= g^{-1}[g(S), h(S)] \\ &= [S, h(S)]. \end{aligned}$$

Hence $[S, h(S)]$ is an n-annulus. Obviously $h(S)$ is collared and $\mathrm{Cl}\ [I(h(S))]$ is an n-cell.

LEMMA 2. *Let S be a collared $(n-1)$-sphere in the interior of an n-cell Q such that* $\mathrm{Cl}\ [I(S)]$ *is an n-cell.*[1] *Suppose M is a compact subset of $\mathring{Q}$. Then there is a collared $(n-1)$-sphere S' in $\mathring{Q}$ such that $I(S')\supset M\cup S$,* $\mathrm{Cl}\ [I(S')]$ *is an n-cell, and $[S, S']$ is an n-annulus.*

PROOF. We may suppose without loss of generality that Q is the unit ball B_1 in E^n and that $I(S)$ contains the origin. Let $\epsilon>0$ be small enough so that $B_\epsilon\subset I(S)$ and $M\cup S\subset \mathring{B}_{1-\epsilon}$. Let h be a homeomorphism of B_1 upon itself such that $h|B_{\epsilon/2}=1$ and $h(\mathring{B}_\epsilon)\supset B_{1-\epsilon}$. Then $S'=h(S)$ contains $M\cup S$ in its interior. Lemma 1 insures that $h(S)$ is collared and that $[S, h(S)]$ is an n-annulus.

THEOREM. *Let X be a topological space which is the union of a sequence $V_1\subset V_2\subset\cdots\subset V_i\subset\cdots$ of open subsets where each V_i is homeomorphic to E^n. Then X is homeomorphic to E^n.*

PROOF. Let h_i map E^n homeomorphically onto V_i. Then $h_1(B_1)$ is an n-cell in V_1. There is an integer n_2 such that

$$h_2(\mathring{B}_{n_2})\supset h_1(B_2)\cup h_2(B_2).$$

Inductively, there is a sequence of integers $n_3, n_4, \cdots$, such that for all i,

$$h_i(\mathring{B}_{n_i})\supset h_1(B_i)\cup\cdots\cup h_i(B_i)\cup h_{n_{i-1}}(B_{i-1}).$$

Since X is locally Euclidean, $h_i(B_{n_i})$ is an n-cell in X containing $h_{i-1}(B_{n_{i-1}})$ in its interior $h_i(\mathring{B}_{n_i})$. Finally $\bigcup_{i=1}^\infty B_{n_i}=X$. For if $x\in X$ there is an integer j such that $x\in V_j$. Hence there is an integer $k>j$ such that $x\in h_j(B_k)$. But then $x\in h_k(B_{n_k})$. Let $Q_i=h_i(B_{n_i})$. Then $X=\bigcup_1^\infty Q_i$ where Q_i is an n-cell, $Q_i\subset\mathring{Q}_{i+1}$, and $\mathring{Q}_{i+1}$ is open in X.

Let S_1 be a collared $(n-1)$-sphere in $\mathring{Q}_1$ such that $\mathrm{Cl}\ [I(S_1)]$ is an n-cell. Applying Lemma 2 to the n-cell Q_2, we obtain a collared $(n-1)$-sphere S_2 in $\mathring{Q}_2$ such that $I(S_2)\supset Q_1\cup S_1$, $[S_1, S_2]$ is an n-annulus, and $\mathrm{Cl}\ [I(S_2)]$ is an n-cell. The same lemma applied to Q_3 and S_2 yields us a collared sphere S_3 in $\mathring{Q}_3$ such that $I(S_3)\supset S_2\cup Q_2$, $[S_2, S_3]$ is an n-annulus, and $\mathrm{Cl}\ [I(S_3)]$ is an n-cell. Continuing this argument, we get a sequence $S_1, S_2, \cdots$, of $(n-1)$-spheres such that $[S_i, S_{i+1}]$ is an n-annulus and $X=I(S_1)\cup[S_1, S_2]\cup[S_3, S_4]\cup \cdots$. Evidently X is homeomorphic to E^n.

References

1. Morton Brown, *A proof of the generalized Schoenflies theorem*, Bull. Amer. Math. Soc. vol. 66 (1960) pp. 74–76.

2. Barry Mazur, *Embeddings of spheres*, Bull. Amer. Math. Soc. vol. 65 (1959) p. 65.

3. Ronald Rosen, *A weak form of the star conjecture for manifolds*, Abstract 570-28, Notices Amer. Math. Soc. vol. 7 (1960) p. 380.

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THREADS WITHOUT IDEMPOTENTS

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If a thread S has no idempotents and if $S^2=S$, then S is iseomorphic with the real interval (0, 1) under ordinary multiplication [2, Corollary 5.6]. Although the result is not nearly as pleasing as the special case just quoted, we shall give here a description of any thread without idempotents. Recall from [1] that a thread is a connected topological semigroup in which the topology is that induced by a total order.

First some examples. Let X be a totally ordered set which is a connected space in the interval topology, let T be a subset of X containing, with t, all elements less than t, and let ϕ be any continuous function from X into (0, 1) whose restriction, ϕ_0, to T is a strictly order-preserving map of T onto $(0, a^2)$ where $a=\text{l.u.b. } \phi(X)$. (We admit that a might be 1.) For such a ϕ to exist it is evidently necessary that X not have a least element, that T not have a greatest element and, provided $T\neq X$ so that the least upper bound, q, of T exists, that $\phi(q)=a^2$.

If $\phi(X)$ is the open interval $(0, a)$, define a multiplication in X by: $x\circ y=\phi_0^{-1}(\phi(x)\phi(y))$. With this definition it is quite easy to see that X is a thread without idempotents and that ϕ is a homomorphism.

In the event that $\phi(X)$ is the half closed interval $(0, a]$ (which implies of course that $a<1$), put $A=\phi^{-1}(a)$ and $B=\phi^{-1}(a^2)$, observe that q must be the least element of B, and let ψ be any continuous

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