THE MONOTONE UNION OF OPEN $n$-CELLS IS AN OPEN $n$-CELL

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In a research announcement [2] B. Mazur indicated that modulo the Generalized Schoenflies Theorem, the following theorem could be proved:

"If the open cone over a topological space $X$ is locally Euclidean at the origin, then it is topologically equivalent with Euclidean space."

Ronald Rosen [3] has described an ingenious proof of this theorem based on the now known [1] Generalized Schoenflies Theorem. In the present paper we prove a stronger theorem without employing the Generalized Schoenflies Theorem.

Definitions and notation. If $Q$ is an $n$-cell then $\hat{Q}$, $\check{Q}$ denote the interior and boundary of $Q$, respectively. An $n$-annulus is a homeomorph of $S^{n-1} \times [01]$. If $S$ is an $(n-1)$-sphere in an $n$-cell, then $I(S)$ denotes the interior (complementary domain) of $S$. If $S_1$, $S_2$ are $(n-1)$-spheres in an $n$-cell and $S_1 \subset I(S_2)$, then $[S_1, S_2]$ (or equivalently $[S_2, S_1]$) denotes the set $Cl[I(S_2)] - I(S_1)$. An $(n-1)$-sphere $S$ embedded in a space $X$ is collared if there is a homeomorphism $h$ of $S^{n-1} \times [01]$ into $X$ such that $h(S^{n-1} \times 1/2) = S$. Finally $B_t$ will denote the $n$-ball of radius $t$ in $E^n$ and centered at the origin.

Lemma 1. Let $S$ be a collared $(n-1)$-sphere in the interior of an $n$-cell $Q$ such that $Cl[I(S)]$ is an $n$-cell. Let $h$ be a homeomorphism of $Q$ upon itself such that $S \subset I(h(S))$ and $h|U = 1$ where $U$ is a nonempty open subset of $I(S)$. Then $h(S)$ is a collared $(n-1)$-sphere in $Q$, $Cl[I(h(S))]$ is an $n$-cell and $[S, h(S)]$ is an $n$-annulus.

Proof. Let $f$ be a homeomorphism of $S^{n-1} \times [01]$ into $\hat{Q}$ such that $f(S^{n-1} \times 0) = S$, $f(S^{n-1} \times [01]) \cap h(S) = 0$, and $f(S^{n-1} \times [01]) \cap I(S) = 0$. Evidently $I(S) \cup f(S^{n-1} \times [01])$ is an $n$-cell. Hence there is a homeomorphism $g$ of $Q$ upon itself such that:

\begin{align*}
(1) & \quad g(S) \subset U, \\
(2) & \quad g|f(S^{n-1} \times 1/2) = S, \\
(3) & \quad g|h(S) = 1.
\end{align*}

Then

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1 The results of [1] make this last part of the hypothesis unnecessary.
Hence \([S, h(S)]\) is an \(n\)-annulus. Obviously \(h(S)\) is collared and \(\text{Cl} [I(h(S))]\) is an \(n\)-cell.

**Lemma 2.** Let \(S\) be a collared \((n-1)\)-sphere in the interior of an \(n\)-cell \(Q\) such that \(\text{Cl} [I(S)]\) is an \(n\)-cell. Suppose \(M\) is a compact subset of \(\hat{Q}\). Then there is a collared \((n-1)\)-sphere \(S'\) in \(\hat{Q}\) such that \(I(S) \supset M \cup S\), \(\text{Cl} [I(S')]\) is an \(n\)-cell, and \([S, S']\) is an \(n\)-annulus.

**Proof.** We may suppose without loss of generality that \(Q\) is the unit ball \(B_1\) in \(E^n\) and that \(I(S)\) contains the origin. Let \(\varepsilon > 0\) be small enough so that \(B_1 \subset I(S)\) and \(M \subset I(S)\). Let \(h\) be a homeomorphism of \(B_1\) upon itself such that \(h|B_1/2 = I\) and \(h(B_1) \supset B_1 - \varepsilon\). Then \(S' = h(S)\) contains \(M \cup S\) in its interior. Lemma 1 insures that \(h(S)\) is collared and that \([S, h(S)]\) is an \(n\)-annulus.

**Theorem.** Let \(X\) be a topological space which is the union of a sequence \(V_1 \subset V_2 \subset \cdots \subset V_i \subset \cdots\) of open subsets where each \(V_i\) is homeomorphic to \(E^n\). Then \(X\) is homeomorphic to \(E^n\).

**Proof.** Let \(h_i\) map \(E^n\) homeomorphically onto \(V_i\). Then \(h_i(B_1)\) is an \(n\)-cell in \(V_i\). There is an integer \(n_2\) such that

\[
h_2(\hat{B}_{n_2}) \supset h_1(B_2) \cup h_2(B_2).
\]

Inductively, there is a sequence of integers \(n_3, n_4, \ldots\), such that for all \(i\),

\[
h_i(\hat{B}_{n_i}) \supset h_1(B_i) \cup \cdots \cup h_i(B_i) \cup h_{i-1}(B_{i-1}).
\]

Since \(X\) is locally Euclidean, \(h_i(B_{n_i})\) is an \(n\)-cell in \(X\) containing \(h_{i-1}(B_{n_{i-1}})\) in its interior \(h_i(\hat{B}_{n_i})\). Finally \(\bigcup_{i=1}^\infty B_{n_i} = X\). For if \(x \in X\) there is an integer \(j\) such that \(x \in V_j\). Hence there is an integer \(k > j\) such that \(x \in h_j(B_k)\). But then \(x \in h_k(B_n)\). Let \(Q_i = h_i(B_{n_i})\). Then \(X = \bigcup_{i=1}^\infty Q_i\) where \(Q_i\) is an \(n\)-cell, \(Q_i \subset \hat{Q}_{i+1}\), and \(\hat{Q}_{i+1}\) is open in \(X\).

Let \(S_1\) be a collared \((n-1)\)-sphere in \(\hat{Q}\) such that \(\text{Cl} [I(S_1)]\) is an \(n\)-cell. Applying Lemma 2 to the \(n\)-cell \(Q_i\), we obtain a collared \((n-1)\)-sphere \(S_i\) in \(\hat{Q}\) such that \(I(S_i) \supset Q_i \cup S_i\), \([S_i, S_i]\) is an \(n\)-annulus, and \(\text{Cl} [I(S_i)]\) is an \(n\)-cell. The same lemma applied to \(Q_i\) and \(S_i\) yields us a collared sphere \(S_i\) in \(\hat{Q}\) such that \(I(S_i) \supset S_i \cup Q_i\), \([S_i, S_i]\) is an \(n\)-annulus, and \(\text{Cl} [I(S_i)]\) is an \(n\)-cell. Continuing this argument, we
get a sequence $S_1, S_2, \ldots$, of $(n-1)$-spheres such that $[S_i, S_{i+1}]$ is an $n$-annulus and $X = I(S_1) \cup [S_1, S_2] \cup [S_3, S_4] \cup \cdots$. Evidently $X$ is homeomorphic to $E^n$.

References


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THREADS WITHOUT IDEMPOTENTS

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If a thread $S$ has no idempotents and if $S^2 = S$, then $S$ is isomorphic with the real interval $(0, 1)$ under ordinary multiplication [2, Corollary 5.6]. Although the result is not nearly as pleasing as the special case just quoted, we shall give here a description of any thread without idempotents. Recall from [1] that a thread is a connected topological semigroup in which the topology is that induced by a total order.

First some examples. Let $X$ be a totally ordered set which is a connected space in the interval topology, let $T$ be a subset of $X$ containing, with $t$, all elements less than $t$, and let $\phi$ be any continuous function from $X$ into $(0, 1)$ whose restriction, $\phi_0$, to $T$ is a strictly order-preserving map of $T$ onto $(0, a^2)$ where $a = \text{l.u.b. } \phi(X)$. (We admit that $a$ might be 1.) For such a $\phi$ to exist it is evidently necessary that $X$ not have a least element, that $T$ not have a greatest element and, provided $T \not= X$ so that the least upper bound, $q$, of $T$ exists, that $\phi(q) = a^2$.

If $\phi(X)$ is the open interval $(0, a)$, define a multiplication in $X$ by: $x \circ y = \phi^{-1}(\phi(x)\phi(y))$. With this definition it is quite easy to see that $X$ is a thread without idempotents and that $\phi$ is a homomorphism.

In the event that $\phi(X)$ is the half closed interval $(0, a]$ (which implies of course that $a < 1$), put $A = \phi^{-1}(a)$ and $B = \phi^{-1}(a^2)$, observe that $q$ must be the least element of $B$, and let $\psi$ be any continuous