INTEGRALITY OF A CERTAIN KIND OF GENUS
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Introduction. The author introduced in [1] a new genus which is a generalization of index and A-genus. In this paper we shall give the complete proof of the integrality of our genus. However the integrality of A-genus has been proved in [2]. The author wishes to express his gratitude to Professor F. Hirzebruch for many valuable suggestions.

1. Let $X^{4k}$ be a compact orientable differentiable $4k$-manifold. We defined the new genus by

$$\sum_{i=0}^{\infty} p_i = \prod_i (1 + \gamma_i),$$

$$\prod_i \frac{\left(\frac{\gamma_i}{\tgh(\gamma_i)}\right)^{1/2}}{(1 + \gamma \tgh^2(\gamma_i)^{1/2})} = \sum_{i=0}^{\infty} \Gamma_i(y, p_1, \ldots, p_k)$$

where $p_i$ denotes the Pontryagin class of the dimension $4i$. The actual values of $\Gamma_i$'s are as follows:

$$\Gamma_0 = 1,$$
$$\Gamma_1(y, p_1) = \left(y + \frac{1}{3}\right) p_1,$$
$$\Gamma_2(y, p_1, p_2) = p_2 y^2 + \frac{1}{3} \left(4p_2 - p_1^2\right) y + \frac{1}{45} \left(7p_2 - p_1^2\right),$$
$$\Gamma_3(y, p_1, p_2, p_3) = p_3 y^3 + \frac{1}{3} \left(6p_3 - p_1 p_2\right) y^2$$
$$+ \frac{1}{15} \left(17p_3 - 8p_1 p_2 + 2p_1^3\right) y$$
$$+ \frac{1}{3^3 \cdot 5 \cdot 7} \left(62p_3 - 13p_1 p_2 + 2p_1^3\right).$$

We put

$$\Gamma_i(y, p_1, \ldots, p_k) = \sum_{j=0}^{i} \Gamma_{ij}(p_1, \ldots, p_k) y^j.$$

Our purpose is to prove that

$$\Gamma_{kj}(p_1, \ldots, p_k)[X^{4k}] = \text{integer}$$

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for every \( k \) and \( j \). It suffices to show that
(i) \( \Gamma_k(\rho_1, \cdots, \rho_k)[X^{4k}] \) does not contain the factor 2 in its denominator when it is written as a quotient of relative prime integers.
(ii) \( 2^{\alpha}\Gamma_k(\rho_1, \cdots, \rho_k)[X^{4k}] \) becomes an integer for a suitable integer \( \alpha \).

First of all let us prove (i). It is well known that

\[
\left(1.6\right) \quad \frac{z^{1/2}}{\tanh z} = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k}}{(2k)!} B_k z^k \quad [3, \text{p.}13]
\]

and

\[
\left(1.7\right) \quad \tanh z = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n}(2^{2n} - 1)}{(2n)!} B_n z^{2n-1}.
\]

Moreover it is known that

a. \( B_k \) (Bernoulli number) contains the factor 2 exactly to the first power in its denominator.
b. \( (2k)! \) is not divisible by \( 2^{2k} \) \([2, \text{p.}341]\).

The statement (i) easily follows from these facts and (1.2).

2. Next let us prove (ii). It suffices to show this for the complex algebraic manifold \([4]\). In this case we have

\[
\left(2.1\right) \quad \sum_{i=0}^{\infty} \rho_i = \prod_i \left(1 + \delta_i^2\right), \quad \sum_{i=0}^{\infty} c_i = \prod_i \left(1 + \delta_i\right)
\]

and

\[
\sum_{i=0}^{\infty} \Gamma_i(y, \rho_1, \cdots, \rho_i)
= \prod_i \frac{\delta_i}{\tanh \delta_i} \left(1 + y \tanh^2 \delta_i \right)
\]

\[
\left(2.2\right) \quad = \prod_i \frac{\delta_i}{1 - e^{-\delta_i}} \left(\frac{1}{\tanh \delta_i} + y \tanh \delta_i \right) \left(1 - e^{-\delta_i}\right)
= \prod_i \frac{\delta_i}{1 - e^{-\delta_i}} \left[ \frac{1}{2} \left(1 + e^{-2\delta_i}\right) + y \frac{1}{2} \left(1 - e^{-2\delta_i}\right) \right].
\]

Hence we have
\[ \Gamma_k(\rho_1, \cdots, \rho_k) = \kappa_k \left( \prod_i \frac{\delta_i}{1 - e^{-\delta_i}} \prod_i \left[ \frac{1}{2} (1 + e^{-2\delta_i}) \right] \cdot \left\{ 1 + \frac{1 - e^{-\delta_i}}{2} + \left( \frac{1 - e^{-2\delta_i}}{2} \right)^2 + \cdots + \left( \frac{1 - e^{-2\delta_i}}{2} \right)^{2^k} \right\} \right] \]

\[ + \frac{1}{2} y(1 - e^{-2\delta_i})(1 - e^{-\delta_i}) \left\{ 1 + \frac{1 - e^{-2\delta_i}}{2} + \left( \frac{1 - e^{-2\delta_i}}{2} \right)^2 + \cdots + \left( \frac{1 - e^{-2\delta_i}}{2} \right)^{2^{k-2}} \right\} \]

\[ \overset{\text{(2.3)}}{=} \kappa_k \left( \sum_t y^t \sum_{a_1, \cdots, a_m} A_{a_1} \cdots a_m \sum_{s_1, \cdots, s_m} \exp \left( a_1 \delta_{s_1} + \cdots + a_m \delta_{s_m} \right) \right) \times \prod_i \frac{\delta_i}{1 - e^{-\delta_i}} \]

where \( \delta_i \) denotes some rational number which becomes an integer when it is multiplied by some power of 2 and \( a_i \) denotes some integer. Putting

\[ \sum_{s_1, \cdots, s_m} \exp \left( a_1 \delta_{s_1} + \cdots + a_m \delta_{s_m} \right) = \sum_d d_i \]

we see that \( d_i \) is an integral cohomology class, i.e., \( d_i \in H^{2i}(X^{4k}, \mathbb{Z}) \). Hence we have

\[ \Gamma_k(\rho_1, \cdots, \rho_k) [X^{4k}] = \sum_i b_i T(X^{4k}, W_i) = \sum_i b_i \chi(X^{4k}, W_i) \]

where \( b_i \) denotes some rational number which becomes an integer by multiplying some power of 2 and \( T(X^{4k}, W_i) \) denotes the Todd genus with regard to \( W_i \) and \( \chi(X^{4k}, W_i) \) denotes a complex-analytic vector bundle whose Chern class is \( \sum d_i \) and \( \chi(X^{4k}, W_i) \) denotes the Riemann-Roch number with regard to \( W_i \) [3, p.154]. Thus we have proved (ii).

**References**


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