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IMMERSIONS OF ALMOST PARALLELIZABLE MANIFOLDS

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The purpose of this note is to prove the following theorem:
An almost parallelizable n-manifold M can be immersed in Euclidean q-space \( \mathbb{R}^q \) if \( 2q > 3n \).

By immersion \( f: M \to \mathbb{R}^q \) we mean a continuously differentiable map whose Jacobian matrix has rank \( n = \dim M \) at each point. We denote the differential of an immersion \( f \) by \( df \).

A regular homotopy \( f_t: M \to \mathbb{R}^q \) is a homotopy such that each \( f_t \) is an immersion and \( df_t \) is a homotopy of the tangent bundle of \( M \) into \( \mathbb{R}^q \). In this case \( f_0 \) and \( f_1 \) have equivalent normal bundles.

We say \( M \) is almost parallelizable if the tangent bundle of \( M - x \) is trivial, for some \( x \in M \).

To prove the theorem, we first observe that if \( M \) is not compact, or is bounded, then \( M \) is parallelizable, and by [1, 6.3], \( M \) can be immersed in \( \mathbb{R}^{n+1} \subseteq \mathbb{R}^q \). Hence we assume \( M \) is compact and unbounded. Let \( B \) be an \( n \)-ball diffeomorphically embedded in \( M \), with bounding \( (n-1) \) sphere \( S \). Put \( M_0 = M - \text{int} B \). By the remark above, there is an immersion \( f: M_0 \to \mathbb{R}^{n+1} \). We consider \( f \) as an immersion in \( \mathbb{R}^q \), and we deform \( f \) through a regular homotopy near \( S \), keeping \( f|_S \) fixed, so that if \( X \) is a unit tangent vector to \( M \) at point \( x \in S \) pointing into \( M_0 \), then \( df(X) \) is the unit vector \( e = (0, \ldots, 0, 1) \) normal to \( \mathbb{R}^{q-1} \) in \( \mathbb{R}^q \). We still have \( f(S) \subseteq \mathbb{R}^{q-1} \).

Since the immersion \( f \) is regularly homotopic to an immersion \( M \to \mathbb{R}^{n+1} \), the normal bundle of \( f \) is trivial. This enables us to apply a lemma [2, 3.2] of M. Kervaire, which implies that the Smale in-

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variant of $f|S$ vanishes. By [3, E], therefore, there exists an immersion $g: B \to \mathbb{R}^{r-1}$ such that $g|S = f|S$. We consider $g$ as an immersion in $\mathbb{R}^q$, and we deform $g$ through a regular homotopy, so that if $X$ is the vector above, $dg(-X) = -e$. We now define $h: M \to \mathbb{R}^q$ by $h(x) = f(x)$ or $h(x)$, according to whether $x \in M_0$ or $x \in B$. It is clear that $h$ is an immersion, and the theorem is proved.

References


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