IMMERSIONS OF ALMOST PARALLELIZABLE MANIFOLDS

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The purpose of this note is to prove the following theorem:

An almost parallelizable n-manifold M can be immersed in Euclidean q-space R^q if 2q > 3n.

By immersion f: M → R^q we mean a continuously differentiable map whose Jacobian matrix has rank n = dim M at each point. We denote the differential of an immersion f by df.

A regular homotopy f_t: M → R^n is a homotopy such that each f_t is an immersion and df_t is a homotopy of the tangent bundle of M into R^n. In this case f_0 and f_1 have equivalent normal bundles.

We say M is almost parallelizable if the tangent bundle of M - x is trivial, for some x ∈ M.

To prove the theorem, we first observe that if M is not compact, or is bounded, then M is parallelizable, and by [1, 6.3], M can be immersed in R^{n+1} ⊂ R^n. Hence we assume M is compact and unbounded. Let B be an n-ball diffeomorphically embedded in M, with bounding (n - 1) sphere S. Put M_0 = M - int B. By the remark above, there is an immersion f: M_0 → R^{n+1}. We consider f as an immersion in R^n, and we deform f through a regular homotopy near S, keeping f|S fixed, so that if X is a unit tangent vector to M at point x ∈ S pointing into M_0, then df(X) is the unit vector e = (0, ..., 0, 1) normal to R^n in R^{n+1}. We still have f(S) ⊂ R^{n+1}.

Since the immersion f is regularly homotopic to an immersion M → R^{n+1}, the normal bundle of f is trivial. This enables us to apply a lemma [2, 3.2] of M. Kervaire, which implies that the Smale in-

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variant of $f|S$ vanishes. By [3, E], therefore, there exists an immersion $g: B \to \mathbb{R}^{r-1}$ such that $g|S = f|S$. We consider $g$ as an immersion in $\mathbb{R}^{r}$, and we deform $g$ through a regular homotopy, so that if $X$ is the vector above, $dg(-X) = -e$. We now define $h: M \to \mathbb{R}^{q}$ by $h(x) = f(x)$ or $h(x)$, according to whether $x \subseteq M_0$ or $x \subseteq B$. It is clear that $h$ is an immersion, and the theorem is proved.

References


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