SURFACE AREA OF A CONVEX BODY UNDER AFFINE TRANSFORMATIONS

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1. Introduction. Let \( K \) be a nondegenerate convex body in \( E^n \). We will seek requirements on \( K \) such that its surface area is minimal compared to that of its affine transforms of the same volume. This problem has been studied for \( n=2 \) by Behrend [1], Green [2], and Gustin [3].

2. Necessary and sufficient conditions. Let \( S \) denote the surface of \( K \), \( V(K) \) its volume and \( S(K) \) its surface area.

Theorem 1. Necessary and sufficient conditions that a nondegenerate convex body \( K \) in \( E^n \) have minimal surface area among its affine transforms of volume \( V(K) \) are that

\[
\int_S u_i u_j dS_u = 0 \quad \text{for } i \neq j
\]

and

\[
\int_S u_i^2 dS_u = \frac{1}{n} S(K)
\]

for \( i, j = 1, \ldots, n \) where \( (u_1, \ldots, u_n) \) is the outer unit vector normal to \( S \) at the point with surface area element \( dS_u \).

Proof. We need only consider central affine transformations of the type \( (x, y \) are column vectors of length \( n) \)

\[
y = Bx, \quad \det (B) = 1.
\]

Suppose that the convex body \( K_i (i=1, \ldots, n) \) is transformed into \( K_i^* \) by (1), then the linear combination \( \lambda_1 K_1 + \cdots + \lambda_n K_n \) is transformed into \( \lambda_1 K_1^* + \cdots + \lambda_n K_n^* \). Since volume is preserved, it follows from the definition of mixed volumes [4, p. 38] that

\[
V(K, \cdots, K, E) = V(K_1^*, \cdots, K_n^*).
\]

Let \( E \) be an ellipsoid with supporting function \( E(u) \), then

\[
nV(K, \cdots, K, E) = \int_S E(u) dS_u, \quad |u| = 1.
\]
It is clear, then, that our problem is equivalent to keeping $K$ fixed and minimizing $V(K, \cdots, K, E)$ over all ellipsoids with center at the origin and volume
\begin{equation}
\kappa_n = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}.
\end{equation}

To show the existence of a minimizing ellipsoid, let $U$ be a solid sphere, of radius $r > 0$, contained in $K$ and let $\delta$, of length $a$, be a segment of maximum length contained in $E$. By the inclusion property of mixed volumes [4, p. 41, (5)] and the representation [4, p. 45] for the area of the projection of $U$, we have $V(K, \cdots, K, E) \geq V(U, \cdots, U, \delta) = ar^{n-1}n^{-1}\kappa_{n-1}$. Consequently, a sequence of ellipsoids $E_i$ of the above type, for which $V(K, \cdots, K, E_i)$ tends to the g.l.b. of (3), is uniformly bounded and by Blaschke's selection theorem [4, p. 34] there exists a subsequence of the $E_i$ which converges to a convex body $E^*$. The body $E^*$ is necessarily an ellipsoid and, by the continuity of mixed volumes, $E^*$ is a minimizing ellipsoid.

Now, let $A^{-1}$ be a positive definite symmetric matrix with $\det(A^{-1}) = 1$. If $E$ is the ellipsoid with surface
\begin{equation}x^TA^{-1}x = 1
\end{equation}
(the superscript $T$ meaning transpose) then the polar reciprocal of $E$ with respect to the origin has distance function
\begin{equation}E(u) = \left\{u^TAu\right\}^{1/2}.
\end{equation}
The above is clear if $A^{-1}$ is a diagonal matrix and the general case follows by considering an orthogonal transformation. Consequently, (6) is the supporting function of $E$, see [4, p. 28], and if $E$ is transformed into $E^*$ by (1) then $E^*(u)$ is found by replacing $A$ in (6) by $BAB^\tau$.

Let $F(B)$ be the function of $n^2$ variables $b_{ij}$ given by
\begin{equation}F(B) = \int SE_A(u)dS_u, \quad |u| = 1,
\end{equation}
where $E_A(u)$ is given by (6) and $A = BB^\tau$. Since $BB^\tau$ is positive semidefinite, (7) is defined for all $B$. We wish to minimize (7), with the constraint
\begin{equation}\det(B) = 1.
\end{equation}
If we equate to zero the partial derivatives of $F(B) + \lambda(1 - \det(B))$ and use (8) we have

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Also

\[
\left[ \frac{\partial F}{\partial b_{ij}} \right] = \left[ \int_S u_i u_j E_A^{-1}(u) dS_u \right] B.
\]

Consequently,

\[
A^{-1} = \frac{1}{\lambda} \left[ \int_S u_i u_j E_A^{-1}(u) dS_u \right], \quad \lambda = \frac{1}{n} \int_S E_A(u) dS_u.
\]

If \( A \) is any positive definite symmetric matrix with \( \det(A) = 1 \) which satisfies (11) then the ellipsoid with supporting function (6) will be called a critical ellipsoid of \( K \). It is clear that the unit matrix satisfies (11) if and only if the conditions in Theorem 1 are satisfied. Consequently, a proof of the following result completes the proof of Theorem 1.

**Theorem 2.** If \( K \) is a nondegenerate convex body with surface \( S \), then there exists one and only one positive definite symmetric matrix \( A \) with \( \det(A) = 1 \) such that

\[
A^{-1} = \frac{n}{\int_S E_A(u) dS_u} \left[ \int_S u_i u_j E_A^{-1}(u) dS_u \right],
\]

where \( E_A(u) \) is given by (6).

**Proof.** Suppose \( C \) is the matrix of an affine transformation of type (1) which carries \( K \) into \( K^* \). If \( F^*(D) \) is the corresponding function (7) for \( K^* \) then

\[
F(B) = F^*(CB)
\]

for all \( B \). By (2) and (3), this is the case for \( B \) with \( \det(B) = 1 \) and by the special form of (7) this may be extended to all \( B \). We have then

\[
\left[ \frac{\partial F}{\partial b_{ij}} \right] = C^T \left[ \frac{\partial F^*}{\partial d_{ij}} \right], \quad D = CB.
\]

Consequently, by (11), (13), (10) and (12) we conclude that a critical ellipsoid of \( K \) is carried by \( C \) into a critical ellipsoid of \( K^* \). We may, therefore, assume that the solid unit sphere is a critical ellipsoid of \( K \) and consequently that the conditions in Theorem 1 hold. Let
and

\[(14)\]
\[E(a, u) = \left[ \sum_{i=1}^{n} \frac{a_i^2}{u_i^2} \right]^{1/2}\]

and

\[(15)\]
\[F(a) = \frac{1}{S(K)} \int_{S} E(a, u) dS_u.\]

The function \(F(a)\) is defined for all points \(a\) in \(E^n\) and has the following properties:

1. \(F(a) > 0\) for \(a \neq 0\) and \(F(0) = 0\),
2. \(F(\lambda a) = \lambda F(a), \lambda > 0\),
3. \(F(a+b) \leq F(a) + F(b)\).

To prove property (3), we have by Cauchy’s inequality

\[\sum a_i b_i u_i^2 \leq \left( \sum a_i^2 u_i^2 \right) \left( \sum b_i^2 u_i^2 \right)^{1/2}\]

and consequently

\[E(a + b, u) \leq E(a, u) + E(b, u).\]

By [4, pp. 21–22], \(F(a)\) is the distance function of a convex body and, by the conditions in Theorem 1, this convex body lies entirely in the closed half-space containing the origin and bounded by the tangent plane \(\sum x_i/n^{1/2} = n^{1/2}\). Now let \(M\) be the set of all points \(a\) such that \(a_i > 0\) and \(a_1 \cdots a_n = 1\). By the inequality between the geometric and arithmetic means we have \(\sum a_i/n^{1/2} \geq n^{1/2}\) for all points in \(M\) with equality only if \(a_i = 1\) \((i = 1, \cdots, n)\). Consequently, \(F(a)\) evaluated over all points in \(M\) has a unique absolute minimum for \(a = (1, \cdots, 1)\). This completes the proof of Theorem 2 since the conditions in Theorem 1 are invariant under an orthogonal transformation.

3. The projection body. Green [2] has shown that for \(n = 2\) the conditions in Theorem 1 may be replaced by the requirement that the second Fourier coefficients of the supporting function \(H(\cos \theta, \sin \theta) = h(\theta)\) of \(K\) vanish. We may generalize this to \(n\)-dimensions if in place of the supporting function of \(K\) we use the supporting function \(\sigma(u)\) of the projection body of \(K\), see [4, p. 45]. This may be viewed as a generalization since for \(n = 2\), \(\sigma(\cos \theta, \sin \theta) = h(\theta + \pi/2) + h(\theta - \pi/2)\).

**Theorem 3.** If \(K\) is a nondegenerate convex body whose projection body has supporting function \(\sigma(u)\), then necessary and sufficient conditions that the surface area of \(K\) be minimal among its affine transforms of volume \(V(K)\) are that
\[ \int_{\Omega} u_i u_j \sigma(u) d\omega_u = 0 \quad \text{for } i \neq j \]

and

\[ \int_{\Omega} u_i^2 \sigma(u) d\omega_u = \frac{\kappa_{n-1}}{n} S(K), \quad |u| = 1 \]

for \( i, j = 1, \ldots, n \) where the integration is extended over the surface \( \Omega \) of the solid unit sphere.

**Proof.** The supporting function \( \sigma(u) \) is given, see [4, pp. 48–49], by

(16) \[ \sigma(u) = \frac{1}{2} \int_{S} |u \cdot \tau| dS_{\tau}, \quad |\tau| = 1 \]

and

(17) \[ S(K) = \frac{1}{\kappa_{n-1}} \int_{\Omega} \sigma(u) d\omega_u, \quad |u| = 1. \]

Let \( S_2(u) \) be any surface harmonic of degree 2, then by the Funk-Hecke theorem [5, pp. 247–248] we have for unit vectors \( u \) and \( \tau \)

\[ \int_{\Omega} |u \cdot \tau| S_2(u) d\omega_u = \frac{\pi^{(n-1)/2}}{\Gamma((n + 3)/2)} S_2(\tau). \]

Consequently, by (16),

\[ \int_{\Omega} S_2(u) \sigma(u) d\omega_u = \frac{1}{2} \frac{\pi^{(n-1)/2}}{\Gamma((n + 3)/2)} \int_{S} S_2(\tau) dS_{\tau} \]

and Theorem 3 follows directly from Theorem 1 and (17).

**References**