STABILITY FOR INHOMOGENEOUS DIFFERENCE SCHEMES[1]

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PART I

Preliminaries. We consider a PDE

$$u_t - Au = f, \tag{1}$$

where u is a (possibly vector-valued) unknown function of a real "time" variable t and an N-dimensional real vector "space" variable x. Here A is a linear operator, constant[2] in t, operating on u, where u is considered a function of x alone (i.e., A acts on elements of a linear space $\mathfrak{B}$ and, for each value of t, $u(\cdot, t) \in \mathfrak{B}$). The function f is a known function of x and t. Thus u is to satisfy (1) for $t > 0$ and $x \in \mathfrak{R}$. The initial condition

$$u(x, 0) = \bar{u}(x) \qquad (x \in \mathfrak{R}) \tag{2}$$

and the boundary condition

$$\Gamma u(\cdot, t) = g(t) = g(*, t) \qquad (t > 0) \tag{3}$$

are assumed to hold. The operator Γ is a linear transformation from $\mathfrak{B}$ to another linear space $\mathfrak{D}$ of functions $[g(t) \in \mathfrak{D}$ for each $t > 0]$ defined on $\partial\mathfrak{R}$, the boundary of $\mathfrak{R}$, such that Γu depends only on the values of u near the boundary.

We assume this problem is well-posed. That is, we assume the existence, for each fixed $T > 0$, of a one-parameter family of operators $\hat{S}_t$ acting on a space Σ with elements of the form

$$w = \begin{pmatrix} \bar{u} \\ \gamma \\ \phi \end{pmatrix},$$

where $\bar{u}$ is a function[3] of $x \in \mathfrak{R}$, γ is a function of $x \in \partial\mathfrak{R}$ and $0 < t$

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[2] As in [1], this is not a necessary requirement but is imposed only to avoid notational clumsiness. The modification necessary to the general case should be clear. The operator Γ appearing in equation (3) may also depend on t.

[3] The values of $\bar{u}$ and ϕ will be the same sort of vectors as u and f; those of γ will be as g.

$\leqq T$, and ϕ is a function of $x \in \mathcal{R}$ and $0 < t \leqq T$. We also assume that each $w \in \Sigma$ satisfies

$$\Gamma \bar{u} = \gamma(0). \tag{4}$$

The operators $\hat{S}_t$ are to send Σ into $\mathcal{B}$ in such a way that for each $w \in \Sigma$

$$\begin{aligned}
&\text{(a)}\ \|\hat{S}_t w - \bar{u}\| \to 0, && \text{as } t \to 0+,\\
&\text{(b)}\ \Gamma \hat{S}_t w = \gamma(t), && \text{for } 0 < t \leqq T,\\
&\text{(c)}\ \|(\hat{S}_{t+\delta} w - \hat{S}_t w)/\delta - A\hat{S}_t w - \phi(t)\| \to 0, \text{ as } \delta \to 0, && \\
& && \text{for } 0 < t < T,\\
&\text{(d)}\ \|\hat{S}_t\| \leqq M = M(T), && \text{for } 0 < t \leqq T.
\end{aligned} \tag{5}$$

It is convenient to define the translation operators[4] T_t and T_t by

$$\begin{aligned}
{}_{\llcorner}T_t\gamma]_{\llcorner}(s, \tau) &= \begin{cases} \gamma(x, \tau + t), & \tau + t \leqq T,\ x \in \partial\mathcal{R},\\ 0, & \tau + t > T,\ x \in \partial\mathcal{R}; \end{cases}\\
[T_t\phi](x, \tau) &= \begin{cases} \phi(x, \tau + t), & \tau + t \leqq T,\ x \in \mathcal{R},\\ 0, & \tau + t > T,\ x \in \mathcal{R}. \end{cases}
\end{aligned} \tag{6}$$

Clearly, $\hat{S}_t$ can be written as

$$\hat{S}_t = (U_t, G_t, F_t), \tag{7}$$

where

$$U_t\bar{u} = \hat{S}_t \begin{pmatrix} \bar{u} \\ 0 \\ 0 \end{pmatrix}, \qquad G_t\gamma = \hat{S}_t \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix}, \qquad F_t\phi = \hat{S}_t \begin{pmatrix} 0 \\ 0 \\ \phi \end{pmatrix},$$

and it is convenient to define the semi-group $\{S_t\}$ by

$$S_t = \begin{pmatrix} U_t & G_t & F_t \\ 0 & T_t & 0 \\ 0 & 0 & T_t \end{pmatrix}. \tag{8}$$

It follows from (5) that $\{S_t\}$ is a bounded, strongly continuous[5] semi-group, and provides the solution of the problem determined by (1), (2), and (3).

[4] There should be no confusion caused by using the same letter for these two different operators, distinguishing them only by their arguments.

[5] We assume that ϕ and γ are restricted so that the operators T_t are strongly continuous. For example, we may assume them to be continuous in t if the maximum norm is used. Actually, a finite number of discontinuities may be handled also without difficulty.

If we now consider, for each $\delta>0$, a mesh μ_δ consisting of a finite number of points in $\mathcal{R}$, each taken at $t=0, \delta, 2\delta, \cdots$, then (sufficiently smooth) functions in Σ may be approximated by functions determined by their values on μ_δ. We identify the approximating functions with functions on μ_δ in such a way as to approximate norms.

By a difference scheme, we mean a family of operators $\tilde{S}_\delta$ such that $(\tilde{S}_\delta)^n$ approximates $S_{n\delta}$ in an appropriate sense. We will write $\tilde{S}_\delta$ in the form

$$\tilde{S}_\delta = \begin{pmatrix} \tilde{u}_\delta & \tilde{G}_\delta & \tilde{F}_\delta \\ 0 & \tilde{T}_\delta & 0 \\ 0 & 0 & \tilde{T}_\delta \end{pmatrix}, \tag{9}$$

and, if $\tilde{w}$ is the function on μ_δ corresponding to an arbitrary but sufficiently smooth function $w\in\Sigma$ of the type described above [let $\tilde{w}_1=\tilde{u}$, $\tilde{w}_2=\tilde{\gamma}$, $\tilde{w}_3=\tilde{\phi}$, where $\tilde{w}=(\tilde{u}, \tilde{\gamma}, \tilde{\phi})^T$], we will ask that

$$\|[(\tilde{S}_\delta\tilde{w})_1 - \tilde{u}]/\delta - Au - \phi(0)\| \to 0 \qquad \text{as } \delta\to 0. \tag{10}$$

If (10) is satisfied, we say that $\tilde{S}_\delta$ is *consistent* with the problem. If there is a constant $\tilde{M}=\tilde{M}(T)$ such that

$$\|\tilde{S}_\delta{}^n\| \leqq \tilde{M} \qquad \text{for } 0\leqq n\delta\leqq T, \tag{11}$$

then we say the scheme is *stable*. If $\tilde{S}_\delta{}^n$ converges strongly to S_t as $\delta\to 0$ with $(n\delta)\to t$, we say the scheme is *convergent*.

Convergence implies stability. The proof here is identical with that in [1].

Suppose the scheme is convergent. Then, since $[0, T]$ is closed, either $\|\tilde{S}_\delta{}^n w_0\|$ is uniformly bounded for each $w_0\in\Sigma$ and $n\delta\leqq T$, or, for some fixed $t_0\in[0, T]$ and fixed $w_0\in\Sigma$, there would be a sequence $\delta_j\to 0$ and a sequence n_j such that $n_j\delta_j\to t_0$ for which $\|(S_{\delta_j})^{n_j}\tilde{w}_0\|$ becomes arbitrarily large. This, however, would contradict the convergence; thus, for each $w_0\in\Sigma$ there is an $m=m(T, w_0)$ such that $\|\tilde{S}_\delta{}^n w_0\|\leqq m$ for $0\leqq n\delta\leqq T$. Since, for each δ, $\tilde{S}_\delta$ is effectively finite-dimensional, it is obviously bounded as is $\tilde{S}_\delta{}^n$. Hence, by the Principle of Uniform Boundedness, the operators $\tilde{S}_\delta{}^n$ are uniformly bounded for $n\delta\leqq T$ and the scheme is stable.

Stability implies convergence. This again is substantially as presented in [1].

Let $w_0\in\Sigma$ and $w(t)=S_t w_0$. From (5) the "first component" of $w(t)$ is the desired solution of the problem. By (5c), we have, for any $\epsilon>0$, a δ_1 such that, for $0<\delta<\delta_1$,

$$(12)\qquad \|[\hat{S}_\delta w(t) - u(t)]/\delta - Au(t) - \phi(t)\| \leqq \epsilon/4\tilde{M}T \quad \text{for } 0 \leqq t < T.$$

Assuming the scheme $\{\tilde{S}_\delta\}$ is stable, we have by (10), with $w = w(t)$, a δ_2 such that, for $0 < \delta < \delta_2$,

$$(13)\qquad \|[(\tilde{S}_\delta \tilde{w}(t))_1 - \tilde{u}(t)]/\delta - Au(t) - \phi(t)\| \leqq \epsilon/4\tilde{M}T \quad \text{for } 0 \leqq t < T.$$

Note further that for w sufficiently smooth there is a δ_3 such that for $0 < \delta < \delta_3$

$$(14)\qquad \|\tilde{u} - u\| \leqq \epsilon/4\tilde{M}T \qquad \text{for } 0 \leqq t \leqq T.$$

It follows from (11), (12), and (13) that for $0 < \delta < \delta_1, \delta_2, \delta_3$

$$(15)\qquad \|(\tilde{S}_\delta \tilde{w}(t))_1 - \hat{S}_\delta w(t)\| \leqq 3\epsilon\delta/4\tilde{M}T \qquad \text{for } 0 \leqq t \leqq T.$$

Let $\delta_j \to 0+$, and n_j be such that $n_j\delta_j \to t \in [0, T]$. Set

$$(16)\qquad \begin{aligned} \psi_j &= [(\tilde{S}_{\delta_j})^{n_j}\tilde{w}_0]_1 - \hat{S}_{n_j\delta_j} w_0 \\ &= \sum_{k=0}^{n_j-1} [\tilde{S}^k_{\delta_j}(\tilde{S}_{\delta_j} - S_{\delta_j}) S_{(n_j-1-k)\delta_j} w_0]_1. \end{aligned}$$

Then, using the stability condition and (15), with $t = (n_j - 1 - k)\delta_j$, we have, for $\delta_j < \delta_0 = \min\{\delta_1, \delta_2, \delta_3\}$,

$$(17)\qquad \|\psi_j\| \leqq \sum_0^{n_j-1} \tilde{M}(3\epsilon\delta_j/4\tilde{M}T) = \frac{3}{4}(n_j\delta_j/T)\epsilon \leqq \frac{3}{4}\epsilon.$$

Putting $s = |n_j\delta_j - t|$, we have

$$\pm(S_{n_j\delta_j} - S_t)w_0 = S_{t'}(S_s - I)w_0,$$

where the sign on the left denotes whether $n_j\delta_j$ is greater than or less than t, and where t' is the smaller of $n_j\delta_j$ and t. Therefore,

$$(18)\qquad \|\hat{S}_{n_j\delta_j} w_0 - \hat{S}_t w_0\| \leqq \|S_{t'}\| \cdot \|S_s w_0 - w_0\|,$$

which goes to zero as $s \to 0$ (i.e., as $j \to \infty$), according to (5d) and (5a). Thus, for any $\epsilon > 0$, there is a J_1 such that for $j > J_1$

$$(19)\qquad \|\hat{S}_{n_j\delta_j} w_0 - \hat{S}_t w_0\| \leqq \epsilon/4.$$

Letting J be greater than J_1 and also large enough so that $\delta_j < \delta_0$ for $j > J$, we have, then, from (15), (17), and (19),

$$(20)\qquad \begin{aligned} \|(\tilde{S}^{n_j}_{\delta_j}\tilde{w}_0)_1 - \hat{S}_t w_0\| &= \|\psi_j + \hat{S}_{n_j\delta_j} w_0 - \hat{S}_t w_0\| \\ &\leqq 3\epsilon/4 + \epsilon/4 = \epsilon. \end{aligned}$$

Thus, since ϵ was arbitrary, we have shown that $(\tilde{S}_{\delta_j})^{n_j}$ converges

strongly to S_t as $j \to \infty$, and, since the sequence $\{\delta_j\}$ was arbitrary, this proves convergence.

Part II

The concept of stability can be generalized to boundary value problems of the form:

$$\begin{aligned} Lu &= f, \\ \Gamma u &= \gamma, \end{aligned} \tag{21}$$

where L is a linear differential operator acting on functions $u = u(x)$ for x in a domain $\mathfrak{D}$ and Γ is a linear boundary operator mapping u into a function γ defined on $\partial\mathfrak{D}$.

If the problem is well-posed, there is a bounded linear operator S such that

$$S\begin{pmatrix} f \\ \gamma \end{pmatrix} = u \tag{22}$$

gives the solution u of (21). Clearly S is the inverse of

$$A = \begin{pmatrix} L \\ \Gamma \end{pmatrix}.$$

By a difference scheme we mean a parametrized sequence of meshes μ_δ becoming dense as $\delta \to 0$ and a family of operators $\tilde{S}_\delta$ intended[6] to approximate S as $\delta \to 0$.

We say the scheme $\{\tilde{S}_\delta\}$ is *consistent* with (21) if for every sufficiently smooth function u

$$\|Au - \tilde{S}_\delta^{-1}\tilde{u}\| \to 0 \qquad \text{as } \delta \to 0. \tag{23}$$

The scheme is called *stable* if the operators $\tilde{S}_\delta$ are uniformly bounded so there is a constant M such that

$$\|\tilde{S}_\delta \tilde{w}\| \leqq M\|\tilde{w}\| \leqq M'(\|f\| + \|\gamma\|). \tag{24}$$

Finally, the scheme is *convergent* if $\tilde{S}_\delta$ converges strongly to S as $\delta \to 0$ so that for a dense set of w

$$\|\tilde{S}_\delta \tilde{w} - Sw\| \to 0 \qquad \text{as } \delta \to 0. \tag{25}$$

H. Keller observed[7] that it is extremely simple to show that stability implies convergence in this situation also. Ignoring the distinction

[6] In practice, of course, it is $\tilde{S}_\delta^{-1}$ which is given.

[7] Private communication. This will appear in the forthcoming book on numerical analysis by E. Isaacson and H. Keller.

between such functions as u and $\bar{u}$ (the representative function agreeing with u on μ_δ and approaching u in the norm) the proof runs as follows:

From (23), there is, for any $\epsilon>0$, a δ_0 such that, for $\delta<\delta_0$, $\|w-S_\delta^{-1}u\|<\epsilon$ where we write $Au=w$, $Sw=u$. On multiplying $w'=(w-\tilde{S}_\delta^{-1}u)$ by $\tilde{S}_\delta$ one has, from (24) $\|\tilde{S}_\delta w-u\|=\|S_\delta w'\|\leqq M'\|w'\|<M'\epsilon$ which gives (25) and the scheme is convergent.

As in Part I, the converse follows from the Principle of Uniform Boundedness.

Reference

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DUALITY IN HOMOGENEOUS PROGRAMMING[1]

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The problem of maximizing a concave function subject to linear constraints does not have a dual, as is the case in linear programming, in which primal optimizing variables do not appear. As a special case of our principal result it will follow that such a dual does indeed exist whenever the objective function is also homogeneous.

In the linear case we are given an $m\times n$ matrix A and vectors $a\in R^n$, $b\in R^m$.[2] The feasibility sets X and Y are defined by: $X=R_+^m\cap\{x\mid xA\leqq a\}$, $Y=R_+^n\cap\{y\mid Ay\geqq b\}$. Since $xA\leqq a$ if and only if $xAy\leqq ay$ for all $y\in R_+^n$ (and similarly for $Ay\geqq b$), we may write:

$$
\begin{aligned}
X &= R_+^m\cap\{x\mid xAy\leqq\psi(y)\quad \text{all } y\in R_+^n\}\\
Y &= R_+^n\cap\{y\mid xAy\geqq\phi(x)\quad \text{all } x\in R_+^m\}
\end{aligned}
\tag{1}
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[2] R^m denotes the set of all real m-tuples. If $u, v\in R^m$ then $u\leqq v$ means that the inequality holds for each component. In particular, $R_+^m=R^m\cap\{x\mid x\geqq 0\}$. If M is a $p\times q$ matrix and N is a $q\times t$ matrix then MN represents the usual matrix product. To simplify notation, the same symbol is used for both a column vector and its transpose; the meaning will, in any case, be clear from the context.