STABILITY FOR INHOMOGENEOUS DIFFERENCE SCHEMES
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Part I

Preliminaries. We consider a PDE

\[ u_t - Au = f, \]

where \( u \) is a (possibly vector-valued) unknown function of a real "time" variable \( t \) and an \( N \)-dimensional real vector "space" variable \( x \). Here \( A \) is a linear operator, constant in \( t \), operating on \( u \), where \( u \) is considered a function of \( x \) alone (i.e., \( A \) acts on elements of a linear space \( \mathfrak{B} \)) and, for each value of \( t \), \( u(\cdot, t) \in \mathfrak{B} \). The function \( f \) is a known function of \( x \) and \( t \). Thus \( u \) is to satisfy (1) for \( t > 0 \) and \( x \in \mathbb{R} \). The initial condition

\[ u(x, 0) = \bar{u}(x) \quad (x \in \mathbb{R}) \]

and the boundary condition

\[ \Gamma u(\cdot, t) = g(t) = g(*, t) \quad (t > 0) \]

are assumed to hold. The operator \( \Gamma \) is a linear transformation from \( \mathfrak{B} \) to another linear space \( \mathfrak{D} \) of functions \( g(t) \in \mathfrak{D} \) for each \( t > 0 \) defined on \( \partial \mathfrak{B} \), the boundary of \( \mathfrak{B} \), such that \( \Gamma u \) depends only on the values of \( u \) near the boundary.

We assume this problem is well-posed. That is, we assume the existence, for each fixed \( T > 0 \), of a one-parameter family of operators \( \mathcal{S}_t \), acting on a space \( \Sigma \) with elements of the form

\[ w = \begin{bmatrix} \bar{u} \\ \gamma \\ \phi \end{bmatrix}, \]

where \( \bar{u} \) is a function\(^3\) of \( x \in \mathfrak{B} \), \( \gamma \) is a function of \( x \in \partial \mathfrak{B} \) and \( 0 < t < T \).

\( ^1 \) This work was performed under the auspices of the U. S. Atomic Energy Commission.

\( ^2 \) As in [1], this is not a necessary requirement but is imposed only to avoid notational clumsiness. The modification necessary to the general case should be clear. The operator \( \Gamma \) appearing in equation (3) may also depend on \( t \).

\( ^3 \) The values of \( \bar{u} \) and \( \phi \) will be the same sort of vectors as \( u \) and \( f \); those of \( \gamma \) will be as \( g \).

Received by the editors September 21, 1960.
\[ T_i w = \gamma (t), \quad \text{for } 0 < t < T, \]
\[ \frac{\| (T_{i+\delta} w - T_i w)/\delta - A T_i w - \phi (t) \|}{\delta} \rightarrow 0, \quad \text{as } \delta \rightarrow 0, \]
\[ \| T_i \| \leq M = M(T), \quad \text{for } 0 < t \leq T. \]

It is convenient to define the translation operators\(^\#\) \( T_t \) and \( T_t \) by

\[ \begin{align*}
[T_t y](x, t) &= \gamma (x, t + t), \quad \tau + t \leq T, \ x \in \partial \Omega, \\
&T_t \phi (x, t) = \phi (x, t + t), \quad \tau + t > T, \ x \in \Omega.
\end{align*} \]

Clearly, \( \hat{S}_t \) can be written as

\[ \hat{S}_t = (U_t, G_t, F_t), \]

where

\[ U_t \hat{u} = \hat{S}_t \begin{pmatrix} \hat{u} \\ \phi \end{pmatrix}, \quad G_t \gamma = \hat{S}_t \begin{pmatrix} 0 \\ \gamma \end{pmatrix}, \quad F_t \phi = \hat{S}_t \begin{pmatrix} 0 \\ \phi \end{pmatrix}, \]

and it is convenient to define the semi-group \( \{ S_t \} \) by

\[ S_t = \begin{pmatrix} U_t & G_t & F_t \\
0 & T_t & 0 \\
0 & 0 & T_t \end{pmatrix}. \]

It follows from (5) that \( \{ S_t \} \) is a bounded, strongly continuous\(^\#\) semi-group, and provides the solution of the problem determined by (1), (2), and (3).

\( ^\#\) There should be no confusion caused by using the same letter for these two different operators, distinguishing them only by their arguments.

\( ^\#\) We assume that \( \phi \) and \( \gamma \) are restricted so that the operators \( T_t \) are strongly continuous. For example, we may assume them to be continuous in \( t \) if the maximum norm is used. Actually, a finite number of discontinuities may be handled also without difficulty.
If we now consider, for each $\delta > 0$, a mesh $\mu_\delta$ consisting of a finite number of points in $\mathfrak{S}$, each taken at $t = 0, \delta, 2\delta, \cdots$, then (sufficiently smooth) functions in $\Sigma$ may be approximated by functions determined by their values on $\mu_\delta$. We identify the approximating functions with functions on $\mu_\delta$ in such a way as to approximate norms. By a difference scheme, we mean a family of operators $\mathfrak{S}_\delta$ such that $(\mathfrak{S}_\delta)^n$ approximates $S_{n\delta}$ in an appropriate sense. We will write $\mathfrak{S}_\delta$ in the form

$$
\mathfrak{S}_\delta = \begin{pmatrix}
\bar{u}_\delta & \bar{C}_\delta & \bar{F}_\delta \\
0 & \bar{T}_\delta & 0 \\
0 & 0 & \bar{T}_\delta
\end{pmatrix},
$$

and, if $\bar{w}$ is the function on $\mu_\delta$ corresponding to an arbitrary but sufficiently smooth function $w \in \Sigma$ of the type described above [let $\bar{w}_1 = \bar{u}$, $\bar{w}_2 = \bar{v}$, $\bar{w}_3 = \bar{\phi}$, where $\bar{w} = (\bar{u}, \bar{v}, \bar{\phi})^T$], we will ask that

$$
\left\| \left[ (\mathfrak{S}_\delta \bar{w})_1 - \bar{u} \right] / \delta - A \bar{u} - \phi(0) \right\| \to 0 \quad \text{as } \delta \to 0.
$$

If (10) is satisfied, we say that $\mathfrak{S}_\delta$ is consistent with the problem. If there is a constant $\bar{M} = \bar{M}(T)$ such that

$$
\left\| (\mathfrak{S}_\delta^n \bar{w}) \right\| \leq \bar{M}
$$

for $0 \leq n\delta \leq T$, then we say the scheme is stable. If $\mathfrak{S}_\delta^n$ converges strongly to $S_t$ as $\delta \to 0$ with $(n\delta) \to t$, we say the scheme is convergent.

Convergence implies stability. The proof here is identical with that in [1].

Suppose the scheme is convergent. Then, since $[0, T]$ is closed, either $\left\| \mathfrak{S}_\delta^n w_0 \right\|$ is uniformly bounded for each $w_0 \in \Sigma$ and $n\delta \leq T$, or, for some fixed $t_0 \in [0, T]$ and fixed $w_0 \in \Sigma$, there would be a sequence $\delta_i \to 0$ and a sequence $n_i$ such that $n_i \delta_i \to t_0$ for which $\left\| (\mathfrak{S}_\delta)^n \bar{w}_0 \right\|$ becomes arbitrarily large. This, however, would contradict the convergence; thus, for each $w_0 \in \Sigma$ there is an $m = m(T, w_0)$ such that $\left\| \mathfrak{S}_\delta^n w_0 \right\| \leq m$ for $0 \leq n\delta \leq T$. Since, for each $\delta$, $\mathfrak{S}_\delta$ is effectively finite-dimensional, it is obviously bounded as is $\mathfrak{S}_\delta^n$. Hence, by the Principle of Uniform Boundedness, the operators $\mathfrak{S}_\delta^n$ are uniformly bounded for $n\delta \leq T$ and the scheme is stable.

Stability implies convergence. This again is substantially as presented in [1].

Let $w_0 \in \Sigma$ and $w(t) = S_tw_0$. From (5) the "first component" of $w(t)$ is the desired solution of the problem. By (5c), we have, for any $\epsilon > 0$, a $\delta_1$ such that, for $0 < \delta < \delta_1$, 
(12) \[ \| [S_{t}w(t) - u(t)]/\delta - Au(t) - \phi(t) \| \leq \varepsilon /4MT \quad \text{for } 0 \leq t < T. \]

Assuming the scheme \( \{ S_{i} \} \) is stable, we have by (10), with \( w = w(t) \), a \( \delta_{2} \) such that, for \( 0 < \delta < \delta_{2} \),

(13) \[ \| [(S_{t}w(t))_{i} - u(t)]/\delta - Au(t) - \phi(t) \| \leq \varepsilon /4MT \quad \text{for } 0 \leq t < T. \]

Note further that for \( w \) sufficiently smooth there is a \( \delta_{3} \) such that for \( 0 < \delta < \delta_{3} \)

(14) \[ \| \bar{u} - u \| \leq \varepsilon /4MT \quad \text{for } 0 \leq t \leq T. \]

It follows from (11), (12), and (13) that for \( 0 < \delta < \delta_{1}, \delta_{2}, \delta_{3} \)

(15) \[ \| (S_{t}w(t))_{i} - S_{t}w(t) \| \leq 3\varepsilon /4MT \quad \text{for } 0 \leq t \leq T. \]

Let \( \delta_{j} \rightarrow 0+ \), and \( n_{j} \) be such that \( n_{j}\delta_{j} \rightarrow t \in [0, T] \). Set

\[ \psi_{j} = [(S_{t})^{n_{j}}w_{0}]_{i} - S_{n_{j}\delta_{j}}w_{0} \]

(16)

\[ = \sum_{k=0}^{n_{j}-1} [S_{k}(S_{t} - S_{k})S_{(n_{j}-1-k)\delta_{j}}w_{0}]_{i}. \]

Then, using the stability condition and (15), with \( t = (n_{j} - 1 - k)\delta_{j} \), we have, for \( \delta_{j} < \delta_{0} = \min \{ \delta_{1}, \delta_{2}, \delta_{3} \} \),

(17) \[ \| \psi_{j} \| \leq \sum_{k=0}^{n_{j}-1} M(3\varepsilon /4MT) = \frac{3}{4} (n_{j}\delta_{j}/T) \varepsilon \leq \frac{3}{4} \varepsilon. \]

Putting \( s = |n_{j}\delta_{j} - t| \), we have

\[ \pm (S_{n_{j}\delta_{j}} - S_{t})w_{0} = S_{t}t(S_{t} - t)w_{0}, \]

where the sign on the left denotes whether \( n_{j}\delta_{j} \) is greater than or less than \( t \), and where \( t' \) is the smaller of \( n_{j}\delta_{j} \) and \( t \). Therefore,

(18) \[ \| S_{n_{j}\delta_{j}}w_{0} - S_{t}w_{0} \| \leq \| S_{t}t \| \cdot \| S_{t}w_{0} - w_{0} \|, \]

which goes to zero as \( s \rightarrow 0 \) (i.e., as \( j \rightarrow \infty \)), according to (5d) and (5a). Thus, for any \( \varepsilon > 0 \), there is a \( J_{1} \) such that for \( j > J_{1} \)

(19) \[ \| S_{n_{j}\delta_{j}}w_{0} - S_{t}w_{0} \| \leq \varepsilon /4. \]

Letting \( J \) be greater than \( J_{1} \) and also large enough so that \( \delta_{j} < \delta_{0} \) for \( j > J \), we have, then, from (15), (17), and (19),

(20) \[ \| (S_{t}^{n_{j}}w_{0})_{i} - S_{t}w_{0} \| = \| \psi_{j} + S_{n_{j}\delta_{j}}w_{0} - S_{t}w_{0} \| \leq 3\varepsilon /4 + \varepsilon /4 = \varepsilon. \]

Thus, since \( \varepsilon \) was arbitrary, we have shown that \( (S_{t})^{n_{j}} \), converges
strongly to $S_t$ as $j \to \infty$, and, since the sequence $\{\delta_j\}$ was arbitrary, this proves convergence.

**Part II**

The concept of stability can be generalized to boundary value problems of the form:

\begin{align}
Lu &= f, \\
\Gamma u &= \gamma,
\end{align}

where $L$ is a linear differential operator acting on functions $u = u(x)$ for $x$ in a domain $\mathcal{D}$ and $\Gamma$ is a linear boundary operator mapping $u$ into a function $\gamma$ defined on $\partial \mathcal{D}$.

If the problem is well-posed, there is a bounded linear operator $S$ such that

\begin{align}
S \begin{pmatrix} f \\ \gamma \end{pmatrix} &= u
\end{align}

gives the solution $u$ of (21). Clearly $S$ is the inverse of $A = \begin{pmatrix} L \\ \Gamma \end{pmatrix}$.

By a difference scheme we mean a parametrized sequence of meshes $\mu_\delta$ becoming dense as $\delta \to 0$ and a family of operators $\tilde{S}_\delta$ intended to approximate $S$ as $\delta \to 0$.

We say the scheme $\{\tilde{S}_\delta\}$ is **consistent** with (21) if for every sufficiently smooth function $u$

\begin{align}
\|Au - \tilde{S}_\delta^{-1}u\| \to 0 \quad \text{as } \delta \to 0.
\end{align}

The scheme is called **stable** if the operators $\tilde{S}_\delta$ are uniformly bounded so there is a constant $M$ such that

\begin{align}
\|\tilde{S}_\delta w\| \leq M\|w\| \leq M'(\|f\| + \|\gamma\|).
\end{align}

Finally, the scheme is **convergent** if $\tilde{S}_\delta$ converges strongly to $S$ as $\delta \to 0$ so that for a dense set of $w$

\begin{align}
\|\tilde{S}_\delta w - Sw\| \to 0 \quad \text{as } \delta \to 0.
\end{align}

H. Keller observed\(^7\) that it is extremely simple to show that stability implies convergence in this situation also. Ignoring the distinction

\(^6\) In practice, of course, it is $\tilde{S}_\delta^{-1}$ which is given.

\(^7\) Private communication. This will appear in the forthcoming book on numerical analysis by E. Isaacson and H. Keller.
between such functions as \( u \) and \( \bar{u} \) (the representative function agreeing with \( u \) on \( \mu_3 \) and approaching \( u \) in the norm) the proof runs as follows:

From (23), there is, for any \( \epsilon > 0 \), a \( \delta_0 \) such that, for \( \delta < \delta_0 \),
\[
\| w - S_\delta^{-1}u \| < \epsilon
\]
where we write \( Au = w, Sw = u \). On multiplying \( w' = (w - S_\delta^{-1}u) \) by \( S_\delta \) one has, from (24)
\[
\| S_\delta w' - u \| = \| S_\delta w' \| \leq M'\| w' \| < M'\epsilon
\]
which gives (25) and the scheme is convergent.

As in Part I, the converse follows from the Principle of Uniform Boundedness.

**Reference**


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**DUALITY IN HOMOGENEOUS PROGRAMMING**

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The problem of maximizing a concave function subject to linear constraints does not have a dual, as is the case in linear programming, in which primal optimizing variables do not appear. As a special case of our principal result it will follow that such a dual does indeed exist whenever the objective function is also homogeneous.

In the linear case we are given an \( m \times n \) matrix \( A \) and vectors \( a \in \mathbb{R}^n, b \in \mathbb{R}^m \). The feasibility sets \( X \) and \( Y \) are defined by:
\[
X = \mathbb{R}^n_+ \cap \{ x \mid xA \leq a \}, \quad Y = \mathbb{R}^m_+ \cap \{ y \mid Ay \geq b \}.
\]
Since \( xA \leq a \) if and only if \( xAy \leq ay \) for all \( y \in \mathbb{R}_+^m \) (and similarly for \( Ay \geq b \)), we may write:
\[
X = \mathbb{R}^n_+ \cap \{ x \mid xAy \leq \psi(y) \} \quad \text{all } y \in \mathbb{R}^m_+
\]
\[
Y = \mathbb{R}^m_+ \cap \{ y \mid xAy \geq \phi(x) \} \quad \text{all } x \in \mathbb{R}^n_+
\]

Received by the editors April 26, 1960; and in revised form, July 27, 1960 and September 12, 1960.

1 Work on this paper was supported, in part, by the Logistics Branch of the Office of Naval Research under Contract NONR 562(15) at Brown University.

2 \( \mathbb{R}^m \) denotes the set of all real \( m \)-tuples. If \( u, v \in \mathbb{R}^m \) then \( u \preceq v \) means that the inequality holds for each component. In particular, \( \mathbb{R}^n_+ = \mathbb{R}_+ \cap \{ x \mid x \geq 0 \} \). If \( M \) is a \( p \times q \) matrix and \( N \) is a \( q \times t \) matrix then \( MN \) represents the usual matrix product.

To simplify notation, the same symbol is used for both a column vector and its transpose; the meaning will, in any case, be clear from the context.

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