STABILITY FOR INHOMOGENEOUS DIFFERENCE SCHEMES\textsuperscript{1}

\textbf{T. SEIDMAN}

\textbf{Part I}

\textbf{Preliminaries.} We consider a PDE

\begin{equation}
\frac{\partial u}{\partial t} - Au = f,
\end{equation}

where $u$ is a (possibly vector-valued) unknown function of a real "time" variable $t$ and an $N$-dimensional real vector "space" variable $x$. Here $A$ is a linear operator, constant\textsuperscript{2} in $t$, operating on $u$, where $u$ is considered a function of $x$ alone (i.e., $A$ acts on elements of a linear space $\mathcal{B}$ and, for each value of $t$, $u(\cdot, t) \in \mathcal{B}$). The function $f$ is a known function of $x$ and $t$. Thus $u$ is to satisfy (1) for $t > 0$ and $x \in \mathcal{R}$. The initial condition

\begin{equation}
u(x, 0) = \bar{u}(x) \quad (x \in \mathcal{R})\end{equation}

and the boundary condition

\begin{equation}
\Gamma u(\cdot, t) = g(t) = g(\ast, t) \quad (t > 0)
\end{equation}

are assumed to hold. The operator $\Gamma$ is a linear transformation from $\mathcal{B}$ to another linear space $\mathcal{D}$ of functions $[g(t) \in \mathcal{D}$ for each $t > 0]$ defined on $\partial \mathcal{R}$, the boundary of $\mathcal{R}$, such that $\Gamma u$ depends only on the values of $u$ near the boundary.

We assume this problem is well-posed. That is, we assume the existence, for each fixed $T > 0$, of a one-parameter family of operators $\mathcal{S}_t$ acting on a space $\Sigma$ with elements of the form

\begin{equation}
w = \begin{bmatrix} \bar{u} \\ \gamma \\ \phi \end{bmatrix},
\end{equation}

where $\bar{u}$ is a function\textsuperscript{3} of $x \in \mathcal{R}$, $\gamma$ is a function of $x \in \partial \mathcal{R}$ and $0 < t$

\textsuperscript{1} This work was performed under the auspices of the U. S. Atomic Energy Commission.

\textsuperscript{2} As in [1], this is not a necessary requirement but is imposed only to avoid notational clumsiness. The modification necessary to the general case should be clear. The operator $\Gamma$ appearing in equation (3) may also depend on $t$.

\textsuperscript{3} The values of $\bar{u}$ and $\phi$ will be the same sort of vectors as $u$ and $f$; those of $\gamma$ will be as $g$. 

778
\[ \Gamma \bar{u} = \gamma(0). \]

The operators $\mathcal{S}_t$ are to send $\Sigma$ into $\mathcal{X}$ in such a way that for each $w \in \Sigma$

(a) $\|\mathcal{S}_t w - \bar{u}\| \to 0$, as $t \to 0^+$,
(b) $\Gamma \mathcal{S}_t w = \gamma(t)$, for $0 < t \leq T$,
(c) $\| (\mathcal{S}_{t+\delta} w - \mathcal{S}_t w)/\delta - \mathcal{A} \mathcal{S}_t w - \phi(\cdot) \| \to 0$, as $\delta \to 0$, for $0 < t < T$,
(d) $\|\mathcal{S}_t\| \leq M = M(T)$, for $0 < t \leq T$.

It is convenient to define the translation operators $T_t$ and $T_\tau$ by

\[ [T_\tau \gamma](x, \tau) = \begin{cases} \gamma(x, \tau + t), & \tau + t \leq T, x \in \partial \mathcal{R}, \\ 0, & \tau + t > T, x \in \partial \mathcal{R}; \end{cases} \]
\[ [T_\tau \phi](x, \tau) = \begin{cases} \phi(x, \tau + t), & \tau + t \leq T, x \in \mathcal{R}, \\ 0, & \tau + t > T, x \in \mathcal{R}. \end{cases} \]

Clearly, $\mathcal{S}_t$ can be written as

\[ \mathcal{S}_t = (U_t, G_t, F_t), \]

where

\[ U_t \bar{u} = \mathcal{S}_t \begin{bmatrix} \bar{u} \\ 0 \end{bmatrix}, \quad G_t \gamma = \mathcal{S}_t \begin{bmatrix} 0 \\ \gamma \end{bmatrix}, \quad F_t \phi = \mathcal{S}_t \begin{bmatrix} 0 \\ \phi \end{bmatrix}, \]

and it is convenient to define the semi-group $\{S_t\}$ by

\[ S_t = \begin{bmatrix} U_t & G_t & F_t \\ 0 & T_t & 0 \\ 0 & 0 & T_t \end{bmatrix}. \]

It follows from (5) that $\{S_t\}$ is a bounded, strongly continuous semi-group, and provides the solution of the problem determined by (1), (2), and (3).

---

4 There should be no confusion caused by using the same letter for these two different operators, distinguishing them only by their arguments.

6 We assume that $\phi$ and $\gamma$ are restricted so that the operators $T_t$ are strongly continuous. For example, we may assume them to be continuous in $t$ if the maximum norm is used. Actually, a finite number of discontinuities may be handled also without difficulty.
If we now consider, for each $\delta > 0$, a mesh $\mu_\delta$ consisting of a finite number of points in $\mathcal{A}$, each taken at $t = 0, \delta, 2\delta, \ldots$, then (sufficiently smooth) functions in $\Sigma$ may be approximated by functions determined by their values on $\mu_\delta$. We identify the approximating functions with functions on $\mu_\delta$ in such a way as to approximate norms.

By a difference scheme, we mean a family of operators $\mathcal{S}_\delta$ such that $(\mathcal{S}_\delta)^n$ approximates $S_{\delta^k}$ in an appropriate sense. We will write $\mathcal{S}_\delta$ in the form

$$
\mathcal{S}_\delta = \begin{pmatrix}
\mathcal{A}_\delta & \mathcal{B}_\delta & \mathcal{C}_\delta \\
0 & \mathcal{T}_\delta & 0 \\
0 & 0 & \mathcal{T}_\delta
\end{pmatrix},
$$

and, if $\tilde{w}$ is the function on $\mu_\delta$ corresponding to an arbitrary but sufficiently smooth function $w \in \Sigma$ of the type described above [let $\tilde{w}_1 = \bar{u}$, $\tilde{w}_2 = \bar{v}$, $\tilde{w}_3 = \phi$, where $\tilde{w} = (\bar{u}, \bar{v}, \phi)^T$], we will ask that

$$
\left\| \frac{(\mathcal{S}_\delta^k \tilde{w})_1 - \bar{u}}{\delta} - \Delta \bar{u} - \phi(0) \right\| \to 0 \quad \text{as } \delta \to 0.
$$

If (10) is satisfied, we say that $\mathcal{S}_\delta$ is consistent with the problem. If there is a constant $\bar{M} = \bar{M}(T)$ such that

$$
\left\| (\mathcal{S}_\delta^k \tilde{w})_1 \right\| \leq \bar{M} \quad \text{for } 0 \leq n\delta \leq T,
$$

then we say the scheme is stable. If $\mathcal{S}_\delta^n$ converges strongly to $S_\delta$ as $n \to 0$ with $(n\delta) \to t$, we say the scheme is convergent.

Convergence implies stability. The proof here is identical with that in [1].

Suppose the scheme is convergent. Then, since $[0, T]$ is closed, either $\|\mathcal{S}_\delta^n w_0\|$ is uniformly bounded for each $w_0 \in \Sigma$ and $n\delta \leq T$, or, for some fixed $t_0 \in [0, T]$ and fixed $w_0 \in \Sigma$, there would be a sequence $\delta_i \to 0$ and a sequence $n_i$ such that $n_i \delta_i \to t_0$ for which $\|\mathcal{S}_\delta^n w_0\|$ becomes arbitrarily large. This, however, would contradict the convergence; thus, for each $w_0 \in \Sigma$ there is an $m = m(T, w_0)$ such that $\|\mathcal{S}_\delta^n w_0\| \leq m$ for $0 \leq n\delta \leq T$. Since, for each $\delta$, $\mathcal{S}_\delta$ is effectively finite-dimensional, it is obviously bounded as is $\mathcal{S}_\delta^n$. Hence, by the Principle of Uniform Boundedness, the operators $\mathcal{S}_\delta^n$ are uniformly bounded for $n\delta \leq T$ and the scheme is stable.

Stability implies convergence. This again is substantially as presented in [1].

Let $w_0 \in \Sigma$ and $w(t) = S_\delta w_0$. From (5) the "first component" of $w(t)$ is the desired solution of the problem. By (5c), we have, for any $\epsilon > 0$, a $\delta_i$ such that, for $0 < \delta < \delta_i$,
(12) \[ \| (S\delta w(t) - u(t)) / \delta - A u(t) - \phi(t) \| \leq \varepsilon / 4MT \quad \text{for } 0 \leq t < T. \]

Assuming the scheme \( \{ S \delta \} \) is stable, we have by (10), with \( w = w(t) \), a \( \delta_2 \) such that, for \( 0 < \delta < \delta_2 \),

(13) \[ \| (S\delta w(t))_1 - u(t) / \delta - A u(t) - \phi(t) \| \leq \varepsilon / 4MT \quad \text{for } 0 \leq t < T. \]

Note further that for \( w \) sufficiently smooth there is a \( \delta_3 \) such that for \( 0 < \delta < \delta_3 \)

(14) \[ \| \delta - u \| \leq \varepsilon / 4MT \quad \text{for } 0 \leq t \leq T. \]

It follows from (11), (12), and (13) that for \( 0 < \delta < \delta_1, \delta_2, \delta_3 \)

(15) \[ \| (S\delta w(t))_1 - S\delta w(t) \| \leq 3\varepsilon / 4MT \quad \text{for } 0 \leq t \leq T. \]

Let \( \delta_j \to 0^+ \), and \( n_j \) be such that \( n_j \delta_j \to t \in [0, T] \). Set

\[ \psi_j = [(S\delta_j)^{n_j}w_0]_1 - S_{n_j \delta_j}w_0 \]

(16)

\[ = \sum_{k=0}^{n_j-1} (S_{\delta_j} - S_{\delta_j}) (S_{n_j-1-k} \delta_j) w_0. \]

Then, using the stability condition and (15), with \( t = (n_j - 1 - k) \delta_j \), we have, for \( \delta_j < \delta_0 = \min \{ \delta_1, \delta_2, \delta_3 \} \),

(17) \[ \| \psi_j \| \leq \sum_{k=0}^{n_j-1} M (3\varepsilon / 4MT) = \frac{3}{4} (n_j \varepsilon / T) \varepsilon \leq \frac{3}{4} \varepsilon. \]

Putting \( s = n_j \delta_j - t \), we have

\[ \pm (S_{n_j \delta_j} - S_t) w_0 = S_t'(S_s - I) w_0, \]

where the sign on the left denotes whether \( n_j \delta_j \) is greater than or less than \( t \), and where \( t' \) is the smaller of \( n_j \delta_j \) and \( t \). Therefore,

(18) \[ \| S_{n_j \delta_j} w_0 - S_t w_0 \| \leq \| S_t' \| \cdot \| S_s w_0 - w_0 \|, \]

which goes to zero as \( s \to 0 \) (i.e., as \( j \to \infty \)), according to (5d) and (5a). Thus, for any \( \varepsilon > 0 \), there is a \( J_1 \) such that for \( j > J_1 \)

(19) \[ \| S_{n_j \delta_j} w_0 - S_t w_0 \| \leq \varepsilon / 4. \]

Letting \( J \) be greater than \( J_1 \) and also large enough so that \( \delta_j < \delta_0 \) for \( j > J \), we have, then, from (15), (17), and (19),

(20) \[ \| (S_{\delta_j}^j w_0) - S_t w_0 \| = \| \psi_j + S_{n_j \delta_j} w_0 - S_t w_0 \| \leq 3\varepsilon / 4 + \varepsilon / 4 = \varepsilon. \]

Thus, since \( \varepsilon \) was arbitrary, we have shown that \( (S_{\delta_j})^j \), converges.
strongly to $S_t$ as $j \to \infty$, and, since the sequence $\{\delta_j\}$ was arbitrary, this proves convergence.

**Part II**

The concept of stability can be generalized to boundary value problems of the form:

$$Lu = f,$$
$$\Gamma u = \gamma,$$

(21)

where $L$ is a linear differential operator acting on functions $u = u(x)$ for $x$ in a domain $\mathcal{D}$ and $\Gamma$ is a linear boundary operator mapping $u$ into a function $\gamma$ defined on $\partial \mathcal{D}$.

If the problem is well-posed, there is a bounded linear operator $S$ such that

$$S \begin{pmatrix} f \\ \gamma \end{pmatrix} = u$$

(22)

gives the solution $u$ of (21). Clearly $S$ is the inverse of

$$A = \begin{pmatrix} L \\ \Gamma \end{pmatrix}.$$

By a difference scheme we mean a parametrized sequence of meshes $\mu_\delta$ becoming dense as $\delta \to 0$ and a family of operators $\bar{S}_\delta$ intended\textsuperscript{6} to approximate $S$ as $\delta \to 0$.

We say the scheme $\{\bar{S}_\delta\}$ is *consistent* with (21) if for every sufficiently smooth function $u$

$$||Au - \bar{S}_\delta^{-1}u|| \to 0 \quad \text{as} \quad \delta \to 0.$$  
(23)

The scheme is called *stable* if the operators $\bar{S}_\delta$ are uniformly bounded so there is a constant $M$ such that

$$||\bar{S}_\delta w|| \leq M ||w|| \leq M' (||f|| + ||\gamma||).$$
(24)

Finally, the scheme is *convergent* if $\bar{S}_\delta$ converges strongly to $S$ as $\delta \to 0$ so that for a dense set of $w$

$$||\bar{S}_\delta w - Sw|| \to 0 \quad \text{as} \quad \delta \to 0.$$  
(25)

H. Keller observed\textsuperscript{7} that it is extremely simple to show that stability implies convergence in this situation also. Ignoring the distinction

\textsuperscript{6} In practice, of course, it is $\bar{S}_\delta^{-1}$ which is given.

\textsuperscript{7} Private communication. This will appear in the forthcoming book on numerical analysis by E. Isaacson and H. Keller.
between such functions as \( u \) and \( \bar{u} \) (the representative function agreeing with \( u \) on \( \mu \) and approaching \( u \) in the norm) the proof runs as follows:

From (23), there is, for any \( \epsilon > 0 \), a \( \delta_0 \) such that, for \( \delta < \delta_0 \),
\[
\| w - S_\delta^{-1} w \| < \epsilon
\]
where we write \( A u = w \), \( S w = u \). On multiplying \( w' = (w - S_\delta^{-1} w) \) by \( S_\delta \), one has, from (24)
\[
\| S_\delta w - u \| = \| S_\delta w' \| \leq M' \| w' \| < M' \epsilon
\]
which gives (25) and the scheme is convergent.

As in Part I, the converse follows from the Principle of Uniform Boundedness.

**Reference**


**University of California, Livermore**

---

**DUALITY IN HOMOGENEOUS PROGRAMMING**

E. EISENBERG

The problem of maximizing a concave function subject to linear constraints does not have a dual, as is the case in linear programming, in which primal optimizing variables do not appear. As a special case of our principal result it will follow that such a dual does indeed exist whenever the objective function is also homogeneous.

In the linear case we are given an \( m \times n \) matrix \( A \) and vectors \( a \in \mathbb{R}^n \), \( b \in \mathbb{R}^m \). The feasibility sets \( X \) and \( Y \) are defined by:

\[ X = \mathbb{R}^n_+ \cap \{ x \mid xA \leq a \}, \quad Y = \mathbb{R}^n_+ \cap \{ y \mid Ay \geq b \}. \]

Since \( xA \leq a \) if and only if \( xAy \leq ay \) for all \( y \in \mathbb{R}^n_+ \) (and similarly for \( Ay \geq b \)), we may write:

\[
X = \mathbb{R}^n_+ \cap \{ x \mid xAy \leq \psi(y) \quad \text{all } y \in \mathbb{R}^n_+ \}
\]

\[
Y = \mathbb{R}^n_+ \cap \{ y \mid xAy \geq \phi(x) \quad \text{all } x \in \mathbb{R}^n_+ \}
\]

Received by the editors April 26, 1960; and in revised form, July 27, 1960 and September 12, 1960.

1 Work on this paper was supported, in part, by the Logistics Branch of the Office of Naval Research under Contract NONR 562(15) at Brown University.

2 \( \mathbb{R}^n \) denotes the set of all real \( n \)-tuples. If \( u, v \in \mathbb{R}^n \) then \( u \leq v \) means that the inequality holds for each component. In particular, \( \mathbb{R}^n_+ = \mathbb{R}^n \cap \{ x \mid x \geq 0 \} \). If \( M \) is a \( p \times q \) matrix and \( N \) is a \( q \times t \) matrix then \( MN \) represents the usual matrix product.

To simplify notation, the same symbol is used for both a column vector and its transpose; the meaning will, in any case, be clear from the context.