0. Introduction. In [2] the maximal ideal space of the tensor product of two commutative Banach algebras was studied. One of the results was: Let $A_3 = A_1 \otimes_A A_2$ be the "greatest cross-norm" [3; 4] tensor product of two commutative Banach algebras $A_1$ and $A_2$. Let $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ be the corresponding spaces of regular maximal ideals. Then $\mathfrak{M}_3$ and $\mathfrak{M}_1 \times \mathfrak{M}_2$ are "naturally" homeomorphic if the weak* topologies are used in all spaces. In the following, we extend the discussion to the case in which no commutativity is assumed. 2

1. Tensor products. Let $A_1$ and $A_2$ be Banach algebras and let $C$ be the complex number system. We consider [3] a subset $F$ of $C^{A_1 \times A_2}$:

$$F = \left\{ f \mid f \in C^{A_1 \times A_2}, f(0, x_2) = f(x_1, 0) = 0, \sum_{(x_1, x_2)} |f(x_1, x_2)| \|x_1\|_1 \|x_2\|_2 < \infty \right\},$$

where $x_i \in A_i$, $i = 1, 2$. Since each $f$ in $F$ is nonzero on a set that is countable or finite, to each $f$ there corresponds a sequence (finite or infinite) $\{(x_{11}, x_{21}), (x_{12}, x_{22}), \ldots \}$ consisting of just the pairs $(x_1, x_2)$ where $f(x_1, x_2) \neq 0$. Addition of elements of $f$ is defined by addition of functions. Multiplication is defined via a form of convolution: If $f, g \in F$, $f \ast g = h$ is defined by:

$$h(0, x_2) = h(x_1, 0) = 0, \quad h(x_1, x_2) = \sum_{a_1 b_1 = x_1, a_2 b_2 = x_2} f(a_1, a_2) g(b_1, b_2)$$

if $x_1, x_2 \neq 0$. We note:

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2 The author is much indebted to the referee of an earlier draft of this research. At his suggestion, arguments that were too compressed for intelligible reading have been elaborated.

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\[ \sum_{(x_1, x_2)} |h(x_1, x_2)| \|x_1\|_1 \|x_2\|_2 \]

\[ \sum_{(a_1, a_2), (b_1, b_2)} |f(a_1, a_2)| \|g(b_1, b_2)| \|a_1\|_1 \|b_1\|_1 \|a_2\|_2 \|b_2\|_2 \]

\[ \left( \sum_{(a_1, a_2)} |f(a_1, a_2)| \|a_1\|_1 \|a_2\|_2 \right) \left( \sum_{(b_1, b_2)} g(b_1, b_2) \|b_1\|_1 \|b_2\|_2 \right) < \infty. \]

Thus \( h \in F \) and the value of \( h(x_1, x_2) \) is given by an absolutely convergent series. Relative to the operations defined, \( F \) is an algebra over \( C \).

If \( \|f\|_1 \) is taken as \( \sum_{(x_1, x_2)} |f(x_1, x_2)| \|x_1\|_1 \|x_2\|_2 \), then an analog of the proof that \( l_1 \) is a Banach space shows that \( F \) is a Banach space.

The relation \( \|fg\|_1 \leq \|f\|_1 \|g\|_1 \) follows from the computation of the preceding paragraph. Hence \( F \) is a Banach algebra.

In \( F \) let \( I \) be the closed ideal generated by all functions \( f \) of the following forms (\( x_1, x_1', x_2, x_2' \) are arbitrary, \( \alpha \) is a complex number):

\((i) \quad f(x_1, x_2) = f(x_1, x_2') = -f(x_1, x_2 + x_2'), f = 0 \) otherwise;
\((ii) \quad f(x_1, x_2) = f(x_1', x_2) = -f(x_1 + x_1', x_2), f = 0 \) otherwise;
\((iii) \quad f(x_1, x_2) = -\alpha f(\alpha x_1, x_2), f = 0 \) otherwise;
\((iv) \quad f(x_1, x_2) = -\alpha f(x_1, \alpha x_2), f = 0 \) otherwise.

We assert that when \( F/I \) is given its quotient space norm it is isometrically isomorphic to \( A_3 \), the greatest cross-norm tensor product \( A_1 \otimes_c A_2 \) of \( A_1 \) and \( A_2 \). The argument rests on the following statements each of which is simply proved.

(a) In \( F \), the set \( F_0 \) of functions \( f \) that are nonzero on sets that are at most finite is dense in \( F \).

(b) \( F_0 \) is a normed algebra containing the generators of \( I \). Let \( I_0 \) be the closed ideal generated in \( F_0 \) by the generators of \( I \). Then \( I_0 \) is a linear space in \( F_0 \) and \( I_0 \) is dense in \( I \).

(c) For \( f \) in \( F_0 \), the quotient norm of \( f/I_0 \) is precisely the greatest cross-norm of the equivalence class represented by \( f \).

(d) The mapping \( \phi_0 : F_0/I_0 \to A_3 \) defined by: \( \phi_0(f/I_0) = \text{equivalence class represented by } f \) is an isometry onto a dense subset of \( A_3 \). The extension \( \phi \) of \( \phi_0 \) to \( F/I \) is an isometry of \( F/I \) onto \( A_3 \).

In dealing with \( A_3 \) (where the norm will be denoted by \( \| \cdots \|_3 \) we shall often work with representatives of elements in \( F/I \). For any \( f \in F \), the element \( g \) given by: \( g(x_1, x_2) = 1 \) if \( (x_1, x_2) = (x_1', x_2) x_1', x_2) \) and \( g = 0 \) otherwise, satisfies the equation \( f/I = g/I \). Thus, ultimately an element of \( A_3 \) will be symbolized by a formal sum \( \sum_{n=1}^\infty (x_{1n} \otimes x_{2n}) \) where \( \sum_{n=1}^\infty \|x_{1n}\|_1 \|x_{2n}\|_2 < \infty \). Very often we shall omit reference to representatives. For example, we shall say simply "the element
2. $M_1 \times M_2 \rightarrow M_3$. Let $(M_1, M_2)$ be in $M_1 \times M_2$. Define the homomorphism $E_3: A_3 \rightarrow (A_1/M_1) \otimes (A_2/M_2)$ by the formula: 

$$E_3\left(\sum_{n=1}^{\infty} (x_1n \otimes x_2n)\right) = \sum_{n=1}^{\infty} (E_1(x_1n) \otimes E_2(x_2n)),$$

where $E_1$ and $E_2$ are the canonical epimorphisms $E_i: A_i \rightarrow A_i/M_i, \ i = 1, 2$. Let $\| \cdot \|_i, \| \cdot \|_j$ denote the quotient norms in $A_1/M_1$ and $A_2/M_2$. Since these algebras are nontrivial so is $(A_1/M_1) \otimes (A_2/M_2)$ nontrivial. Thus $E_3^{-1}(0) = M_3$ is a proper ideal in $A_3$. If $u_1$ and $u_2$ are identities modulo $M_1$ and $M_2$ then $u = u_1 \otimes u_2$ is an identity modulo $M_3$. Thus $M_3$ is regular. Since the norm $\| \cdot \|_i$ in $E_3(A_3)$ satisfies

$$\| E_3\left(\sum_{n=1}^{\infty} (x_1n \otimes x_2n)\right) \|_j \leq \sum_{n=1}^{\infty} \| E_1(x_1n) \|_i \| E_2(x_2n) \|_j,$$

we see that $E_3$ is norm-decreasing and hence that $E_3$ is continuous. Hence $M_3$ is closed.

If $M_3$ is not maximal, let $M_3 \subset N_3$, a regular maximal ideal. Let $F_3: A_3 \rightarrow A_3/N_3$ be the canonical epimorphism and define $G_1$ and $G_2$ by the formulas

$$G_1(x_1) = E_3(ux_1),$$

$$G_2(x_2) = E_3(x_2u).$$

Then $G_i(A_i) \subset A_3/M_3, \ i = 1, 2$. We prove now

**Lemma 1.** A regular ideal $I_3$ in $A_3$ is a right and left $A_1$- and $A_2$- ideal.

**Proof.** Let $u$ be an identity modulo $I_3$ in $A_3, x_1 \in A_1, v \in I_3$. Then $ux_1v - x_1v \in I_3$ and thus $ux_1v/I_3 = x_1v/I_3 = 0$. But $ux_1v/I_3 = (ux_1/I_3)(v/I_3)$ = $0$. Thus $x_1v \in I_3$. Similarly $vx_1 \in I_3$, and $x_2v, x_2v \in I_3$ for $x_2 \in A_2$.

From Lemma 1 we see that $G_2(x_2y_2) = E_3(x_2y_2u) = E_3(x_2(m_3 + uy_2u)) = E_3(x_2uy_2u) = G_2(x_2)G_2(y_2)$ (where $m_3 \in M_3$). Note also: $\|G_2(x_2)\|_i = \|E_3(x_2y_2u)\|_i \leq \|x_2\|_i \|y_2\|_i \leq \|x_2\|_i$. It is clear now that $G_2$ (and similarly $G_1$) is a continuous homomorphism.

If $(x_1, x_2) \in M_1 \times M_2$ then $ux_1 = (u_1x_1 \otimes u_2) = w$ is such that $E_3(w) = 0$, i.e., $G_1(x_1) = 0$. Thus $M_1 \subset G_1^{-1}(0)$. Similarly, $M_2 \subset G_2^{-1}(0)$. Hence either $M_i = G_i^{-1}(0)$ or $G_i(A_i) = 0, \ i = 1, 2$. Since $G_i(A_i) \neq 0$, we see

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*A* may be regarded as a left and right $A_1$- and $A_2$-module, e.g., $x_1(y_1 \otimes y_2) = (x_1y_1 \otimes y_2), \ \ \ \text{where} \ \ x_1, y_1 \in A_1, y_2 \in A_2.$
If $H_i$ are engendered by $F_3$ as $G_i$ are engendered by $E_3$ we find [2] $M_i = H_i^{-1}(0)$, $i = 1, 2$. Following the argument in [2], we obtain $H_i = \alpha_i G_i \beta_i E_i$, where $\alpha_i$, $\beta_i$ are isometric $C^*$-automorphisms of $A_i/M_i$. Finally $E_3 = (\beta_1 \otimes \beta_2)^{-1} F_3$, whence $E_3(A_3)$ is simple and $M_3$ is maximal.

We remark that $E_3$ is an epimorphism. Indeed, if

$$\sum_{n=1}^{\infty} (E_1(x_{1n}) \otimes E_2(x_{2n})) = w \in (A_1/M_1) \otimes (A_2/M_2),$$

we can choose $(y_{1n}, y_{2n}) \in A_1 \times A_2$ so that $E_i(y_{in}) = E_i(x_{in})$, $i = 1, 2$, $n = 1, 2, \ldots$, and such that $\|y_{1n}\|_1 < \|E_1(x_{1n})\|_1 + 2^{-n}$, $\|y_{2n}\|_2 < \|E_2(x_{2n})\|_2 + 2^{-n}$. (We may and do assume $\|E_1(x_{1n})\|_1 = \|E_2(x_{2n})\|_2$.) For, in each nontrivial term we may use the equivalent representative $\alpha E_1(x_{1n}) \otimes \alpha^{-1} E_2(x_{2n})$ where $\alpha = 1 + (\|E_1(x_{1n})\|_1)(\|E_2(x_{2n})\|_2)^{-1}$. Then if $z = \sum_{n=1}^{\infty} (y_{1n} \otimes y_{2n})$ we see $E_3(z) = w$.

We note (for later use) that $G_1(A_1)$ and $G_2(A_2)$ are commuting subalgebras of $A_3/M_3$. For

$$G_1(x_1)G_2(x_2) = E_3(u_1x_1)E_3(x_2u_2) = E_3(u_1 \otimes u_2)E_3(x_1 \otimes x_2u_2),$$

$$= (E_1(u_1x_1) \otimes E_2(u_2)) \cdot (E_1(x_1) \otimes E_2(x_2u_2)) = E_1(x_1) \otimes E_2(x_2),$$

$$= (E_1(u_1) \otimes E_2(x_2u_2)) \cdot (E_1(u_1x_1) \otimes E_2(u_2)) = E_3(x_2u_2)E_3(u_1x_1),$$

$$= G_2(x_2)G_1(x_1).$$

Hence [1] there is a homomorphism $T: G_1(A_1) \otimes G_2(A_2) \rightarrow A_3/M_3$ given by $T(\sum_{n=1}^{\infty} (G_1(x_{1n}) \otimes G_2(x_{2n}))) = \sum_{n=1}^{\infty} G_1(x_{1n})G_2(x_{2n})$. Since $T(G_1(u_1) \otimes G_2(u_2)) = the identity of A_3/M_3$, and since $G_1(A_1) \otimes G_2(A_2)$ is simple we see $T^{-1}(0) = 0$, i.e., $T$ is a monomorphism. From our definitions it now follows that $T$ is an isomorphism.

Similarly, if $\gamma_i$ are isometric isomorphisms $\gamma_i: G_i(A_i) \rightarrow B_i$, $i = 1, 2$, there is an isomorphism $T': B_1 \otimes B_2 \rightarrow A_3/M_3$ given by

$$T' \left( \sum_{n=1}^{\infty} (y_{1n} \otimes y_{2n}) \right) = T \left( \sum_{n=1}^{\infty} (\gamma_1^{-1}(y_{1n}) \otimes \gamma_2^{-1}(y_{2n})) \right).$$

Finally, the association between pairs $(M_1, M_2)$ and ideals $M_3$ is 1-1. For if $(M_1, M_2) \neq (M_1', M_2')$, e.g., if $M_1 \neq M_1'$, choose $x_1 \in M_1 \setminus M_1'$ and $x_2 \in M_2 \setminus M_2'$, choose $x_2 \in M_2$. Then $E_3(x_1 \otimes x_2) = 0$, and $E_3'(x_1 \otimes x_2) \neq 0$, i.e., $(M_1, M_2)$ and $(M_1', M_2')$ engender distinct epimorphisms $E_3$ and $E_3'$.

3. $M_3 \rightarrow M_1 \times M_2$. Consider the algebras $A_i \otimes_\varepsilon A_j$, obtained by adjoining (if necessary) identities $\varepsilon_i$ to $A_i$, $i = 1, 2$. It is to be noted that $A_i \otimes_\varepsilon A_j$, is in general not the same as $A_{i+j}$, the result of
adjoining (if necessary) an identity \( e_3 \) to \( A_3 \). A typical element \( z \) of \( A_4 \) may be represented by 

\[
a(x_1 \otimes e_2) + e_1 \otimes x_3 + x_1 \otimes e_2 + \sum_{n=1}^{\infty} (x_{1n} \otimes x_{2n})
\]

where \( x_1, x_{1n} \in A_1, x_2, x_{2n} \in A_2, \alpha \in C, \sum_{n=1}^{\infty} ||x_{1n}|| ||x_{2n}||_2 < \infty. \) There is an obvious isometric isomorphic copy of \( A_3 \) in \( A_4 \). We denote the copy of \( A_3 \) by \( A_3 \).

**Lemma 2.** \( A_3 \) is an ideal in \( A_4 \).

**Proof.** Let \( z \) be as above, and let \( w = \sum_{n=1}^{\infty} (w_{1n} \otimes w_{2n}) \in A_3 \). Then 

\[
zw = aw + wx_2 + x_1w + vw
\]

where \( v = \sum_{n=1}^{\infty} (x_{1n} \otimes x_{2n}) \). Since all four summands of \( zw \) are in \( A_3 \) the result follows.

A similar argument employing Lemma 1 shows the following.

**Lemma 3.** A closed ideal \( I_3 \) of \( A_3 \) is a closed ideal in \( A_4 \).

**Lemma 4.** Let \( M_3 \) be a regular maximal ideal of \( A_3 \), \( u \) an identity (in \( A_3 \)) modulo \( M_3 \): \( uA_3 - A_3, A_3u - A_3 \subset M_3 \). In \( A_4 \) let 

\[
M_4^0 = \{ z | uz, uzw, uswu, u(zwu) - zwu, zwu \in M_3 \}, A_4^0 = [x_1 \otimes e_2 + e_1 \otimes x_1] + A_3
\]

Then \( M_4^0 \) is a (regular) ideal in \( A_4 \), \( u \) is an identity (in \( A_4 \)) modulo \( M_4^0 \) and \( M_4^0 \subseteq A_4^0 \).

**Proof.** For \( z \in M_4^0, w \in A_4 \), we see in succession: \( uz, uzw, uswu, u(zwu) - zwu, zwu \in M_3 \). Hence \( M_4^0 \) is a right ideal and by a similar proof is a left ideal.

Writing \( e_4 = e_1 \otimes e_2 \), we see \( u(u - e_4) = u^2 - u \in M_3 \), whence \( u - e_4 \in M_4^0 \).

Since (Lemma 2) \( A_4 \) is an ideal in \( A_4 \), we find that if \( z \in A_4 \) then 

\[
uz, zu \in A_3 \text{ and } (uz - z)u = u(zu) - zu, u(uz - z) = u(uz) - u(z) \in M_3,
\]

whence \( uz - z \in M_4^0 \).

We now consider several cases:

**Case 1.** Assume \( A_1 \) and \( A_2 \) have identities. Define mappings \( F_i: A_i \rightarrow A_3/M_3, i = 1, 2 \) by the (symbolically given) formulas 

\[
F_1(A_1) = (A_1 \otimes e_2)/M_3, F_2(A_2) = (e_1 \otimes A_2)/M_3.
\]

Then \( F_i^{-1}(0) \) are regular ideals \( N_i \) of \( A_i, i = 1, 2 \).

We find the pair associated with \( M_3 \) by treating several subcases.

**Case 1.** \( N_1 \) and \( N_2 \) are both maximal. Then as in §2, let them generate an \( N_3 \subset M_3 \) and let \( G_3 \) be the canonical epimorphism 

\[
G_3: A_3 \rightarrow (A_1/N_1) \otimes (A_2/N_2). \]

We shall show \( N_3 = M_3 \) (whence we shall have shown \( N_3 = M_3 \) since both are maximal ideals). Thus let 

\[
z = \sum_{n=1}^{\infty} (x_{1n} \otimes x_{2n}) \in A_3 \text{ and assume } G_3(z) = 0.
\]

Then 

\[
\sum_{n=1}^{\infty} (F_1(x_{1n}) \otimes F_2(x_{2n})) = 0.
\]

\( F_1(A_1) \) and \( F_2(A_2) \) are commuting subalgebras of \( A_3/M_3 \) and thus 

\[
T: (A_1/N_1) \otimes (A_2/N_2) \rightarrow A_3/M_3, \text{ given by } T(\sum_{n=1}^{\infty} (F_1(x_{1n}) \otimes F_2(x_{2n})))
\]
= \sum_{n=1}^{\infty} F_1(x_{1n})F_2(x_{2n}), \text{ is a homomorphism. (Since } T(F_1(u_1) \otimes F_2(u_2)) = \text{ the identity of } A_3/M_3, \text{ and since } F_1(A_1) \otimes F_2(A_2) \text{ is simple, } T \text{ is a monomorphism.) Hence}

F_3(x) = \sum_{n=1}^{\infty} (F_1(x_{1n})F_2(x_{2n})) = T\left( \sum_{n=1}^{\infty} (F_1(x_{1n}) \otimes F_2(x_{2n})) \right) = T(0) = 0.

Thus \( N_3 \subseteq M_3 \).

**Case 2.** \( N_1 \) is not maximal; \( N_2 \) is maximal. Let \( N_1 \subseteq M_1 \subseteq M_3 \). Then \( M_1 \) and \( N_2 \) engender an \( M'_3 \subset M_3 \) (whence, again, we shall have shown \( M'_3 = M_3 \)). This time let \( G'_3 \) be the canonical epimorphism \( G'_3: A_3 \rightarrow (A_1/M_1) \otimes (A_2/N_2) \), and let \( G'_1, G'_2 \) arise from \( G'_3 \) according to the procedure in \( \S 2 \). Then \( \alpha_2 G'_2 = F_2 \), where \( \alpha_2 \) is an isometric C-automorphism of \( F_2(A_2) \). As indicated in \( \S 2 \), there is an isomorphism \( T: G'_1(A_1) \otimes G'_2(A_2) \rightarrow A_3/M'_3 \). If \( L: A_1/N_1 \rightarrow (A_1/N_1)/(F_1(M_1)) \) is the canonical epimorphism, then by virtue of the "second isomorphism theorem" \( LF_1 = \alpha_1 G'_1 \), where \( \alpha_1 \) is an isometric C-automorphism of \( G'_1(A_1) \). If \( G_3(x) = 0 \), then \( \sum_{n=1}^{\infty} (E_1(x_{1n}) \otimes F_2(x_{2n})) = 0 \), \( (E_1: A_1 \rightarrow A_1/M_1) \). Via C-automorphisms we can conclude \( \sum_{n=1}^{\infty} (G'_1(x_{1n}) \otimes G'_2(x_{2n})) = 0 \) and then that

\[
(\alpha_1 \otimes \alpha_2) \left( \sum_{n=1}^{\infty} (G'_1(x_{1n}) \otimes G'_2(x_{2n})) \right) = \sum_{n=1}^{\infty} (\alpha_1 G'_1(x_{1n}) \otimes \alpha_2 G'_2(x_{2n})) = \sum_{n=1}^{\infty} (LF_1(x_{1n}) \otimes F_2(x_{2n})) = 0.
\]

Remembering that \( \alpha_1 \) and \( \alpha_2 \) are automorphisms and applying \( T \) we find \( \sum_{n=1}^{\infty} LF_1(x_{1n})F_2(x_{2n}) = L_3 \left( \sum_{n=1}^{\infty} F_1(x_{1n})F_2(x_{2n}) \right) = L_3 F_3(x) = 0 \) (where \( L_3: A_3/M_3 \rightarrow A_3/M'_3 \) is defined by the initial identity). Since \( L_3 F_1(u_1)F_2(u_2) \neq 0 \) and since \( A_3/M_3 \) is simple we see \( F_3(x) = 0 \), and so \( M'_3 \subset M_3 \).

**Case 3.** Both \( N_1 \) and \( N_2 \) are not maximal. Argue mutatis mutandis as in Case 2.

**Case II.** Not both \( A_1 \) and \( A_2 \) have identities. Let \( M_4 \) be a regular maximal ideal in \( A_4 \) such that (Lemma 4) \( A_4 \not\supset M_4 \not\supset M_3 \). Let \( M_4 \) engender \( M_{i4} \), regular maximal ideals of \( A_{i4} \), \( i = 1, 2 \) (Case I). If \( M_{i4} \), say, is \( A_1 \) then clearly \( A_4/M_4 = 0 \), whence \( u/M_4 = 0 \), a contradiction. Hence \( M_{i4} \nsubseteq A_4 \), and the regular maximal ideals we seek are \( M_i = M_{i4} \cap A_i, \ i = 1, 2 \).

In every case we get a pair \( (M_1, M_2) \) in \( \mathfrak{M}_1 \times \mathfrak{M}_2 \) and this pair in turn engenders \( M_3 \). Since the association \( \mathfrak{M}_1 \times \mathfrak{M}_2 \rightarrow \mathfrak{M}_3 \) is 1-1, \( (M_1, M_2) \) is uniquely determined by \( M_3 \): \( (M_1, M_2) = i(M_3) \). We have shown \( i \) is 1-1 and that \( i(\mathfrak{M}_3) = \mathfrak{M}_1 \times \mathfrak{M}_2 \).
Theorem. When $hk$-topologies are used throughout, $\bar{t}$ is continuous but not generally bicontinuous.

Proof. We shall show that $\bar{t}$ is closed. Let $F_1$ be closed in $M_1$. Then $F_3 = F_1 \times M_2$ is closed in $M_1 \times M_2$. Let $K_3 = \bar{t}^{-1}(F_3)$. We show that $K_3$ is closed. To this end let $M_3 \supseteq k(K_3)$ and assume $M_3 \subseteq K_3$. Thus if $\bar{t}(M_3) = (M_1, M_2)$, then $M_1 \subseteq F_1$. Hence $M_1 \supseteq F_1$, and there is an $x$ in $k(F_1)$ such that $x/M_1 \neq 0$. But if $y$ is an arbitrary element of $A_2$, then $z = x \otimes y$ in $A_3$ is actually in $k(K_3)$. For if $M_3 = \bar{t}^{-1}(M_1, M_2)$ is in $K_3$, then $M_1$ is in $F_1$ and hence $x/M_1 = 0$, whence $z/M_3 = (x/M_1) \otimes (y/M_2) = 0$. Therefore $z$ is in $M_3$, and since $M_3$ is arbitrary in $K_3$, $z$ is in $k(K_3)$. Thus any such $z$ is in $M_3$. But if we choose $y$ so that $y/M_2 \neq 0$, then $z/M_3 \neq 0$. This contradiction shows that $K_3$ is closed. Similarly, if $F_2$ is closed in $M_2$ then $\bar{t}^{-1}(M_1 \times F_2)$ is closed.

Since any closed set in $M_1 \times M_2$ is of the form

$$\bigcap \{ (F_i \times M_2) \cup (M_1 \times F_2) \}$$

where $F_i$ is closed in $M_i, i = 1, 2$, it follows that $\bar{t}$ is closed and thus that $\bar{t}$ is continuous.

We are indebted to John Lindberg for suggesting the following example showing that $\bar{t}$ need not be continuous.

Example. Let $A_1$ be the commutative Banach algebra of functions analytic in the interior of the unit disc $D_z$ and continuous on the entire unit disc $D_z$. The maximal ideal space of $A_1$ may be identified with $D_z$ although the $hh$-topology constitutes a genuine weakening of the usual topology of $D_z$. In fact, a $hh$-closed set $F$ in $D_z$ has only countably many points in the (usual) interior of $D_z$ unless $F$ is the whole of $D_z$. Let $A_z$ be $C(D_w)$, i.e., the algebra of all continuous functions on $D_w$. Then the maximal ideal space of $A_z$ in the $hh$-topology is homeomorphic with $D_w$ in its usual topology.

In $A_3$ let $v$ be the element $z \otimes 1 - 1 \otimes w$ ($z$ and $w$ here are the usual complex variables). Then in $M_1 \times M_2$ the set $K = \{ (z, w) | z = w \}$ is the hull of $v$, and so $\bar{t}^{-1}(K)$ is closed in the $hh$-topology of $M_3$. Thus $U = M_3 \setminus \bar{t}^{-1}(K)$ is open. We shall show that $\bar{t}(U)$ is not open in $M_1 \times M_2$.

If $\bar{t}(U)$ were open, there would be open sets $U_i$ in $M_i$ such that $U_1 \times U_2 \subseteq \bar{t}(U)$, and $U_z$ could be taken as an ordinary open circle in the interior of $D_z$. Since $U_z = M_3 \setminus \bar{t}^{-1}(K)$ is $hh$-closed in $D_z$, $F_1$ has at most countably many points in the ordinary interior of $D_z$. Let $U_z^\sharp$ be the circle in $D_z$ that consists of the same set of complex numbers as the set comprising $U_z$. Then $U_z^\sharp \setminus F_1 \neq \emptyset$ since $F_1$ meets the ordinary interior of $D_z$ in at most countably many points. But then $U_1 \times U_2 \supseteq (U_z^\sharp \setminus F_1) \times U_2$. Clearly, any point $z$ in $U_z^\sharp \setminus F_1$ corresponds to a point.
$w$ in $U_2$ such that $z = w$, i.e., $(U_2^* \setminus F_i) \times U_2$ meets $K$ and thus $U_1 \times U_2$ cannot lie in the complement $\mathcal{I}(U)$ of $K$.

**Remark.** In the last paragraph of p. 302 and the first two lines of p. 303 [2] the proof given is incorrect since the $u_i$ chosen there vary with the $M_i$. The following amended proof should be substituted.

Let $M_{03}$ be in $\mathfrak{M}_3$ and if $\mathcal{I}(M_{03}) = (M_{01}, M_{02})$, let $N(M_{0i})$ in $\mathfrak{M}_i$ be of the form:

$$N(M_{0i}) = \{ M_i \mid a_{ji}^+(M_{0i}) - a_{ji}^+(M_{03}) < r_i, j = 1, 2, \ldots, J_i \},$$

$i = 1, 2$.

To prove the continuity of $\mathcal{I}$ it suffices to find an $N(M_{03})$ such that $\mathcal{I}(N(M_{03})) \subset N(M_{01}) \times N(M_{02})$. To this end let $u_{0i}$ be identities modulo $M_{0i}, i = 1, 2$ and let

$$N(M_{03}) = \{ M_3 \mid \left| \left( a_{j1} \otimes u_{02} \right) \left( M_{03} \right) - \left( a_{j1} \otimes u_{03} \right) \left( M_{03} \right) \right| < s_k, \quad k = 1, 2, j = 1, 2, \ldots, J_1;$$

$$\left| \left( u_{01} \otimes a_{j2} \right) \left( M_{03} \right) - \left( u_{01} \otimes a_{j2} \right) \left( M_{03} \right) \right| < v_m, \quad m = 1, 2, j = 1, 2, \ldots, J_2;$$

$$\left| \left( u_{01} \otimes u_{02} \right) \left( M_{03} \right) - 1 \right| < w \}.$$

We treat two cases: (1) For all $(j, i) a_{ji}^+(M_{0i}) \neq 0$. Then for appropriate choices of the $s_k, v_m$ and $w$, we can assure that

$$\left| \left( a_{j1} \otimes u_{02} \right)^+ \left( M_{03} \right) - \left( a_{j1} \otimes u_{03} \right)^+ \left( M_{03} \right) \right| = \left| a_{j1}^+(M_{01}) - a_{j1}^+(M_{03}) \right| < r_1$$

and similarly that $\left| a_{ji}^+(M_{02}) - a_{ji}^+(M_{03}) \right| < r_2$, where $\mathcal{I}(M_3) = (M_1, M_2)$. In this case then, $\mathcal{I}(N(M_{03})) \subset N(M_{01}) \times N(M_{02})$. (2) Some $a_{ji}^+(M_{0i}) = 0$. For the $(j, i)$ combinations for which $a_{ji}^+(M_{0i}) \neq 0$, $s_k, v_m$ and $w$ are chosen as in (1). For the other $(j, i)$ combinations, since $\left| u_{0i}^+(M_{0i}) \right|$ is bounded, we see that for suitable $w$ and $M_3$ in $N(M_{03}) \left| u_{0i}^+(M_{0i}) \right|$ is bounded away from $0$ and then for suitable $s_k$ and $v_m \left| a_{ji}^+(M_{0i}) \right|$ is small. The result follows upon a final revised choice of $s_k, v_m$ and $w$.

**Bibliography**


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