0. Introduction. In [2] the maximal ideal space of the tensor product of two commutative Banach algebras was studied. One of the results was: Let $A_3 = A_1 \otimes A_2$ be the "greatest cross-norm" [3; 4] tensor product of two commutative Banach algebras $A_1$ and $A_2$. Let $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ be the corresponding spaces of regular maximal ideals. Then $\mathcal{M}_3$ and $\mathcal{M}_1 \times \mathcal{M}_2$ are "naturally" homeomorphic if the weak* topologies are used in all spaces. In the following, we extend the discussion to the case in which no commutativity is assumed.²

1. Tensor products. Let $A_1$ and $A_2$ be Banach algebras and let $C$ be the complex number system. We consider [3] a subset $F$ of $C^{A_1 \times A_2}$:

$$F = \left\{ f \mid f \in C^{A_1 \times A_2}, f(0, x_2) = f(x_1, 0) = 0, \sum_{(x_1, x_2)} |f(x_1, x_2)| \|x_1\|_1 \|x_2\|_2 < \infty \right\},$$

where $x_i \in A_i$, $i = 1, 2$. Since each $f$ in $F$ is nonzero on a set that is countable or finite, to each $f$ there corresponds a sequence (finite or infinite) $\{(x_{11}, x_{21}), (x_{12}, x_{22}), \cdots \}$ consisting of just the pairs $(x_1, x_2)$ where $f(x_1, x_2) \neq 0$. Addition of elements of $f$ is defined by addition of functions. Multiplication is defined via a form of convolution: If $f, g \in F$, $f * g = h$ is defined by:

$$h(0, x_2) = h(x_1, 0) = 0, \quad h(x_1, x_2) = \sum_{a_1b_1 = x_1; a_2b_2 = x_2} f(a_1, a_2)g(b_1, b_2)$$

if $x_1, x_2 \neq 0$. We note:

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Presented to the Society September 4, 1959; received by the editors December 26, 1959 and, in revised form, September 3, 1960 and May 4, 1961.

¹ This research was supported in part by the United States Air Force through Air Force Office of Scientific Research of the Air Research and Development Command, under contract No. AF 49 (638)-526, and in part by the National Science Foundation under grant NSF-G11048. The author thanks these organizations for their assistance. Reproduction of the paper in whole or in part is permitted for any purposes of the United States Government.

² The author is much indebted to the referee of an earlier draft of this research. At his suggestion, arguments that were too compressed for intelligible reading have been elaborated.

750
\[ \sum_{(x_1, x_2)} |h(x_1, x_2)| \|x_1\|_1 \|x_2\|_2 \]
\[ \leq \sum_{(a_1, a_2), (b_1, b_2)} |f(a_1, a_2)| \|g(b_1, b_2)| \|a_1\|_1 \|b_1\|_1 \|a_2\|_2 \|b_2\|_2 \]
\[ = \left( \sum_{(a_1, a_2)} |f(a_1, a_2)| \|a_1\|_1 \|a_2\|_2 \right) \left( \sum_{(b_1, b_2)} |g(b_1, b_2)| \|b_1\|_1 \|b_2\|_2 \right) < \infty. \]

Thus \( h \in F \) and the value of \( h(x_1, x_2) \) is given by an absolutely convergent series. Relative to the operations defined, \( F \) is an algebra over \( C \).

If \( \|f\|_f \) is taken as \( \sum_{(x_1, x_2)} |f(x_1, x_2)| \|x_1\|_1 \|x_2\|_2 \), then an analog of the proof that \( l_1 \) is a Banach space shows that \( F \) is a Banach space. The relation \( \|f \ast g\|_f \leq \|f\|_f \|g\|_f \) follows from the computation of the preceding paragraph. Hence \( F \) is a Banach algebra.

In \( F \) let \( I \) be the closed ideal generated by all functions \( f \) of one of the following forms (\( x_1, x_1', x_2, x_2' \) are arbitrary, \( \alpha \) is a complex number):

(i) \( f(x_1, x_2) = f(x_1, x_2') = -f(x_1, x_2 + x_2'), f = 0 \) otherwise;
(ii) \( f(x_1, x_2) = f(x_1', x_2) = -f(x_1 + x_1', x_2), f = 0 \) otherwise;
(iii) \( f(x_1, x_2) = -\alpha f(\alpha x_1, x_2), f = 0 \) otherwise;
(iv) \( f(x_1, x_2) = -\alpha f(x_1, \alpha x_2), f = 0 \) otherwise.

We assert that when \( F/I \) is given its quotient space norm it is isometrically isomorphic to \( A_3 \), the greatest cross-norm tensor product \( A_1 \otimes A_2 \) of \( A_1 \) and \( A_2 \). The argument rests on the following statements each of which is simply proved.

(a) In \( F \), the set \( F_0 \) of functions \( f \) that are nonzero on sets that are at most finite is dense in \( F \).

(b) \( F_0 \) is a normed algebra containing the generators of \( I \). Let \( I_0 \) be the closed ideal generated in \( F_0 \) by the generators of \( I \). Then \( I_0 \) is a linear space in \( F_0 \) and \( I_0 \) is dense in \( I \).

(c) For \( f \) in \( F_0 \), the quotient norm of \( f/I_0 \) is precisely the greatest cross-norm of the equivalence class represented by \( f \).

(d) The mapping \( \phi_0: F_0/I_0 \rightarrow A_3 \) defined by: \( \phi_0(f/I_0) = \text{equivalence class represented by } f \) is an isometry onto a dense subset of \( A_3 \). The extension \( \phi \) of \( \phi_0 \) to \( F/I \) is an isometry of \( F/I \) onto \( A_3 \).

In dealing with \( A_3 \) (where the norm will be denoted by \( \| \cdot \cdot \|_3 \)) we shall often work with representatives of elements in \( F/I \). For any \( f \in F \), the element \( g \) given by: \( g(x_1, x_2) = 1 \) if \( (x_1, x_2) = (x_1', x_2') \) and \( g = 0 \) otherwise, satisfies the equation \( f/I = g/I \). Thus, ultimately an element of \( A_3 \) will be symbolized by a formal sum \( \sum_{n=1}^\infty (x_{1n} \otimes x_{2n}) \) where \( \sum_{n=1}^\infty \|x_{1n}\|_1 \|x_{2n}\|_2 < \infty \). Very often we shall omit reference to representatives. For example, we shall say simply “the element
2. \( \mathbb{M}_1 \times \mathbb{M}_2 \rightarrow \mathbb{M}_3 \). Let \((M_1, M_2)\) be in \( \mathbb{M}_1 \times \mathbb{M}_2 \). Define the homomorphism \( E_3 : A_3 \rightarrow (A_1/M_1) \otimes (A_2/M_2) \) by the formula:

\[
E_3(\sum_{n=1}^{\infty} (x_{1n} \otimes x_{2n})) = \sum_{n=1}^{\infty} (E_1(x_{1n}) \otimes E_2(x_{2n})),
\]

where \( E_1 \) and \( E_2 \) are the canonical epimorphisms \( E_i : A_i \rightarrow A_i/M_i, i = 1, 2 \). Let \( \| \cdots \|_1, \| \cdots \|_2 \) denote the quotient norms in \( A_1/M_1 \) and \( A_2/M_2 \). Since these algebras are nontrivial so is \( (A_1/M_1) \otimes (A_2/M_2) \) nontrivial. Thus \( E_3^{-1}(0) = M_3 \) is a proper ideal in \( A_3 \). If \( u_1 \) and \( u_2 \) are identities modulo \( M_1 \) and \( M_2 \) then \( u = u_1 \otimes u_2 \) is an identity modulo \( M_3 \). Thus \( M_3 \) is regular. Since the norm \( \| \cdots \|_3 \) in \( E_3(A_3) \) satisfies

\[
\| E_3 \left( \sum_{n=1}^{\infty} (x_{1n} \otimes x_{2n}) \right) \|_3 \leq \sum_{n=1}^{\infty} \| E_1(x_{1n}) \|_1 \| E_2(x_{2n}) \|_2
\]

we see that \( E_3 \) is norm-decreasing and hence that \( E_3 \) is continuous. Hence \( M_3 \) is closed.

If \( M_3 \) is not maximal, let \( M_3 \subset N_3 \), a regular maximal ideal. Let \( F_3 : A_3 \rightarrow A_3/N_3 \) be the canonical epimorphism and define \( G_1 \) and \( G_2 \) by the formulas

\[
G_1(x_1) = E_3(ux_1),
G_2(x_2) = E_3(x_2u).
\]

Then \( G_i(A_i) \subset A_3/M_3, i = 1, 2 \). We prove now

**Lemma 1.** A regular ideal \( I_3 \) in \( A_3 \) is a right and left \( A_1 \)- and \( A_2 \)-ideal.

**Proof.** Let \( u \) be an identity modulo \( I_3 \) in \( A_3 \), \( x_1 \in A_1, v \in I_3 \). Then \( ux_1v - x_1v \in I_3 \) and thus \( ux_1v/I_3 - x_1v/I_3 = 0 \). But \( ux_1v/I_3 = (ux_1/I_3)(v/I_3) = 0 \). Thus \( x_2v \in I_3 \). Similarly \( vx_1 \in I_3 \), and \( x_2v, vx_2 \in I_3 \) for \( x_2 \in A_2 \).

From Lemma 1 we see that \( G_2(x_2y_2) = E_3(x_2y_2u) = E_3(x_2(m_3 + uy_2u)) = E_3(x_2uy_2) = G_2(x_2)G_2(y_2) \) (where \( m_3 \in M_3 \)). Note also: \( \| G_2(x_1) \|_3 \leq \| x_1 \|_3 \leq k \| x_1 \|_1 \). It is clear now that \( G_2 \) (and similarly \( G_1 \)) is a continuous homomorphism.

If \( (x_1, x_2) \in M_1 \times M_2 \) then \( u x_1 = (u_1 x_1 \otimes u_2) = w \) is such that \( E_2(w) = 0 \), i.e., \( G_1(x_1) = 0 \). Thus \( M_1 \subset G_1^{-1}(0) \). Similarly, \( M_2 \subset G_2^{-1}(0) \). Hence either \( M_i = G_i^{-1}(0) \) or \( G_i(A_i) = 0, i = 1, 2 \). Since \( G_1(u_i) \neq 0, \) we see

\[ A_3 \] may be regarded as a left and right \( A_1 \) and \( A_2 \)-module, e.g., \( x_1(y_1 \otimes y_2) = (x_1y_1 \otimes y_2), \) where \( x_1, y_1 \in A_1, y_2 \in A_2 \).
Grô) = Mi. If 77¿ are engendered by F3 as G¿ are engendered by £3 we find [2] Mi = Hrl(0), i—1, 2. Following the argument in [2], we obtain H¡ = α(G¡ = β¡E¡ where α¡, β¡ are isometric C-automorphisms of A¡/M¡. Finally E3 = (β1 ⊗ β2)−1F3, whence E3(A3) is simple and M3 is maximal.

We remark that E3 is an epimorphism. Indeed, if

\[ \sum_{n=1}^{\infty} (E_1(x_{1n}) \otimes E_2(x_{2n})) = w \subseteq (A_1/M_1) \otimes (A_2/M_2), \]

we can choose \((y_{1n}, y_{2n}) \in A_1 \times A_2\) so that \(E_i(y_{in}) = E_i(x_{in}), i=1, 2, n = 1, 2, \ldots\), and such that \(\|y_{1n}\|_1 < \|E_1(x_{1n})\|_1 + 2^{-n}, \|y_{2n}\|_2 < \|E_2(x_{2n})\|_2 + 2^{-n}\). (We may and do assume \(\|E_1(x_{1n})\|_1 = \|E_2(x_{2n})\|_2\).

For, in each nontrivial term we may use the equivalent representative \(\alpha E_1(x_{1n}) \otimes \alpha^{-1} E_2(x_{2n})\) where \(\alpha = +1 [\|E_1(x_{1n})\|_1 (\|E_2(x_{2n})\|_2^{-1})^{-1/2}]\).

Then if \(\mathbf{z} = \sum_{n=1}^{\infty} (y_{1n} \otimes y_{2n})\) we see \(E_3(\mathbf{z}) = w\).

We note (for later use) that \(G_1(A_1)\) and \(G_2(A_2)\) are commuting sub-algebras of \(A_3/M_3\). For

\[ G_1(x_1)G_2(x_2) = E_3(ux_1)E_3(x_2u) = E_3(u_1x_1 \otimes u_2x_2)E_3(u_1 \otimes u_2x_2), \]

Hence there is a homomorphism \(T: G_1(A_1) \otimes G_2(A_2) \rightarrow A_3/M_3\) given by \(T(\sum_{n=1}^{\infty} (G_1(x_{1n}) \otimes G_2(x_{2n}))) = \sum_{n=1}^{\infty} G_1(x_{1n})G_2(x_{2n})\). Since \(T(G_1(u_1) \otimes G_2(u_2)) = \text{the identity of } A_3/M_3\) and since \(G_1(A_1) \otimes G_2(A_2)\) is simple we see \(T^{-1}(0) = 0\), i.e., \(T\) is a monomorphism. From our definitions it now follows that \(T\) is an isomorphism.

Similarly, if \(\gamma_i\) are isometric isomorphisms \(\gamma_i: G_i(A_i) \rightarrow B_i, i=1, 2\), there is an isomorphism \(T': B_1 \otimes B_2 \rightarrow A_3/M_3\) given by

\[ T'\left(\sum_{n=1}^{\infty} (y_{1n} \otimes y_{2n})\right) = T\left(\sum_{n=1}^{\infty} (\gamma_1^{-1}(y_{1n}) \otimes \gamma_2^{-1}(y_{2n}))\right). \]

Finally, the association between pairs \((M_1, M_2)\) and ideals \(M_3\) is 1-1. For if \((M_1, M_2) \neq (M'_1, M'_2)\), e.g., if \(M_1 \neq M'_1\), choose \(x_1 \in M_1 \setminus M'_1\) and \(x_2 \in M_2 \setminus M'_2\) if \(M_2 \neq M'_2\) (otherwise choose \(x_2 \in M_2\)). Then \(E_3(x_1 \otimes x_2) = 0\), and \(E'_1(x_1 \otimes x_2) \neq 0\), i.e., \((M_1, M_2)\) and \((M'_1, M'_2)\) engender distinct epimorphisms \(E_3\) and \(E'_3\).

3. \(M_3 \rightarrow M_1 \times M_2\). Consider the algebras \(A_{4\xi}\) obtained by adjoining (if necessary) identities \(e_i\) to \(A_i\), \(i=1, 2\). It is to be noted that \(A_4 = A_{4e_1} \otimes A_{2e_2}\) is in general not the same as \(A_{2e_2}\), the result of
adjoining (if necessary) an identity \( e_3 \) to \( A_3 \). A typical element \( z \) of \( A_4 \) may be represented by \( \alpha(x_1 \otimes e_2) + e_1 \otimes x_3 + x_1 \otimes e_2 + \sum_{n=1}^{\infty} (x_{1n} \otimes x_{2n}) \) where \( x_1, x_{1n} \in A_1, x_2, x_{2n} \in A_2, \alpha \in C, \sum_{n=1}^{\infty} \|x_{1n}\| \|x_{2n}\|_2 < \infty \). There is an obvious isometric isomorphic copy of \( A_3 \) in \( A_4 \). We denote the copy of \( A_3 \) by \( A_3 \).

**Lemma 2.** \( A_3 \) is an ideal in \( A_4 \).

**Proof.** Let \( z \) be as above, and let \( w = \sum_{n=1}^{\infty} (w_{1n} \otimes w_{2n}) \in A_3 \). Then \( zw = \alpha w + w_3 + x_1 w + wv \) where \( v = \sum_{n=1}^{\infty} (x_{1n} \otimes x_{2n}) \). Since all four summands of \( zw \) are in \( A_3 \) the result follows.

A similar argument employing Lemma 1 shows the following.

**Lemma 3.** A closed ideal \( I_3 \) of \( A_3 \) is a closed ideal in \( A_4 \).

**Lemma 4.** Let \( M_3 \) be a regular maximal ideal of \( A_3 \), \( u \) an identity (in \( A_3 \)) modulo \( M_3 \), \( uA_3 - A_3, A_3u - A_3 \subseteq M_3 \). In \( A_4 \) let \( M_4^0 = \{ z | uz, uz \in M_3 \} \), \( A_4^0 = [x_1 \otimes e_2 + e_1 \otimes x_2] \otimes A_3 \). Then \( M_4^0 \) is a (regular) ideal in \( A_4 \), \( u \) is an identity (in \( A_4 \)) modulo \( M_4^0 \) and \( M_4^0 \subseteq A_4^0 \).

**Proof.** For \( z \in M_4^0, w \in A_4 \), we see in succession: \( uz, uzw, uzwu, u(zwu) - zwu, zwu \in M_3 \). Hence \( M_4^0 \) is a right ideal and by a similar proof is a left ideal.

Writing \( e_4 = e_1 \otimes e_2 \), we see \( u(u - e_4) = u^2 - u \in M_3 \), whence \( u - e_4 \in M_4^0 \cap A_4^0 \).

Since (Lemma 2) \( A_4 \) is an ideal in \( A_4 \), we find that if \( z \in A_4 \) then \( uz, zu \in A_3 \) and \( (uz - z)u = u(zu) - zu \), \( u(uz - z) = u(uz) - uz \in M_3 \), whence \( uz - z \in M_4^0 \).

We now consider several cases:

Case 1. Assume \( A_1 \) and \( A_2 \) have identities. Define mappings \( F_i: A_i \rightarrow A_3/M_3, i = 1, 2 \) by the (symbolically given) formulas \( F_i(A_i) = (A_i \otimes e_2)/M_3, F_2(A_2) = (e_1 \otimes A_2)/M_3 \). Then \( F_i^{-1}(0) \) are regular ideals \( N_i \) of \( A_i, i = 1, 2 \).

We find the pair associated with \( M_3 \) by treating several subcases.

Case 1. \( N_1 \) and \( N_2 \) are both maximal. Then as in §2, let them engender an \( N_3 \subseteq M_3 \) and let \( G_3 \) be the canonical epimorphism \( G_3: A_3 \rightarrow (A_1/N_1) \otimes_g (A_2/N_2) \). We shall show \( N_3 \subset M_3 \) (whence we shall have shown \( N_3 = M_3 \) since both are maximal ideals). Thus let \( z = \sum_{n=1}^{\infty} (x_{1n} \otimes x_{2n}) \in A_3 \) and assume \( G_3(z) = 0 \). Then

\[
\sum_{n=1}^{\infty} (F_1(x_{1n}) \otimes F_2(x_{2n})) = 0.
\]

\( F_1(A_1) \) and \( F_2(A_2) \) are commuting subalgebras of \( A_3/M_3 \) and thus \( T: (A_1/N_1) \otimes_g (A_2/N_2) \rightarrow A_3/M_3 \), given by \( T(\sum_{n=1}^{\infty} (F_1(x_{1n}) \otimes F_2(x_{2n}))) \)
\[ F_3(z) = \sum_{n=1}^{\infty} (F_1(x_1^n)F_2(x_2^n)) = T\left( \sum_{n=1}^{\infty} (F_1(x_1^n) \otimes F_2(x_2^n)) \right) = T(0) = 0. \]

Thus \( N_3 \subseteq M_3 \).

**Case 2.** \( N_1 \) is not maximal; \( N_2 \) is maximal. Let \( N_1 \subseteq M_1 \subseteq \mathfrak{M}_1 \). Then \( M_1 \) and \( N_2 \) engender an \( M'_1 \subseteq \mathfrak{M}_1 \). We shall show \( M'_1 \subseteq M_3 \) (whence, again, we shall have shown \( M'_1 = M_3 \)). This time let \( G'_1 \) be the canonical epimorphism \( G'_1 : A_1 \to (A_1/M_1) \otimes (A_2/N_2) \), and let \( G'_1, G'_2 \) arise from \( G'_1 \) according to the procedure in §2. Then \( \alpha_2G'_2 = F_2 \), where \( \alpha_2 \) is an isometric \( C \)-automorphism of \( F_2(A_2) \). As indicated in §2, there is an isomorphism \( T: G'_1(A_1) \otimes \alpha_1 G'_2(A_2) \to M'_1 \). If \( L: A_1/N_1 \to (A_1/N_1)/(F_1(M_1)) \) is the canonical epimorphism, then by virtue of the "second isomorphism theorem" \( LF_1 = \alpha_1 G'_1 \), where \( \alpha_1 \) is an isometric \( C \)-automorphism of \( G'_1(A_1) \). If \( G_3(z) = 0 \), then

\[ \sum_{n=1}^{\infty} (E_1(x_1^n) \otimes F_2(x_2^n)) = 0, \quad (E_1: A_1 \to A_1/M_1). \]

Via \( C \)-automorphisms we can conclude

\[ \sum_{n=1}^{\infty} (G'_1(x_1^n) \otimes G'_2(x_2^n)) = 0, \]

\[ \sum_{n=1}^{\infty} (LF_1(x_1^n) \otimes F_2(x_2^n)) = 0. \]

Remembering that \( \alpha_1 \) and \( \alpha_2 \) are automorphisms and applying \( T \) we find

\[ \sum_{n=1}^{\infty} LF_1(x_1^n)F_2(x_2^n) = L_3(\sum_{n=1}^{\infty} F_1(x_1^n)F_2(x_2^n)) = L_3F_3(z) = 0 \]

(where \( L_3: A_3/M_3 \to A_3/M_3 \) is defined by the initial identity). Since \( L_3(F_1(u_1)F_2(u_2)) \neq 0 \) and since \( A_3/M_3 \) is simple we see \( F_3(z) = 0 \), and so \( M'_1 \subseteq M_3 \).

**Case 3.** Both \( N_1 \) and \( N_2 \) are not maximal. Argue mutatis mutandis as in Case 2.

**Case II.** Not both \( A_1 \) and \( A_2 \) have identities. Let \( M_4 \) be a regular maximal ideal in \( A_4 \) such that \( (\text{Lemma 4}) A_4 \supseteq M_4 \supseteq A_3 \). Let \( M_4 \) engender \( M_{i_{t_1}} \), regular maximal ideals of \( A_{i_{t_1}}, \ t = 1, 2 \) (Case 1). If \( M_{i_{t_1}}, \) say, is \( A_1 \) then clearly \( A_3/M_4 = 0 \), whence \( u/M_4 = 0 \), a contradiction. Hence \( M_{i_{t_1}} \subseteq A_1 \), and the regular maximal ideals we seek are

\[ M_i = M_{i_{t_1}} \cap A_i, \quad i = 1, 2. \]

In every case we get a pair \( (M_1, M_2) \) in \( \mathfrak{M}_1 \times \mathfrak{M}_2 \) and this pair in turn engenders \( M_3 \). Since the association \( \mathfrak{M}_1 \times \mathfrak{M}_2 \to \mathfrak{M}_3 \) is 1-1, \( (M_1, M_2) \) is uniquely determined by \( M_3 \): \( (M_1, M_2) = \bar{i}(M_3) \). We have shown \( \bar{i} \) is 1-1 and that \( i(\mathfrak{M}_3) = \mathfrak{M}_1 \times \mathfrak{M}_2 \).
Theorem. When $hk$-topologies are used throughout, $\tilde{I}$ is continuous but not generally bicontinuous.

Proof. We shall show that $\tilde{I}^{-1}$ is closed. Let $F_1$ be closed in $M_1$. Then $F_3 = F_1 \times M_2$ is closed in $M_1 \times M_2$. Let $K_3 = \tilde{I}^{-1}(F_3)$. We show that $K_3$ is closed. To this end let $M_3 \supset k(K_3)$ and assume $M_3 \notin K_3$. Thus if $\tilde{I}(M_3) = (M_1, M_2)$, then $M_1 \notin F_1$. Hence $M_1 \notin F_3$ and there is an $x$ in $k(F_1)$ such that $x/M_1 \neq 0$. But if $y$ is an arbitrary element of $A_2$, then $z = x \otimes y$ in $A_3$ is actually in $k(K_3)$. For if $M_3 = \tilde{I}^{-1}(M_1, M_2)$ is in $K_3$, then $M_1$ is in $F_1$ and hence $x/M_1 = 0$, whence $z/M_3 = (x/M_1) \otimes (y/M_2) = 0$. Therefore $z$ is in $M_3$, and since $M_3$ is arbitrary in $K_3$, $z$ is in $k(K_3)$. Thus any such $z$ is in $M_3$. But if we choose $y$ so that $y/M_2 \neq 0$, then $z/M_3 \neq 0$. This contradiction shows that $K_3$ is closed. Similarly, if $E_2$ is closed in $M_2$ then $\tilde{I}^{-1}(M_1 \times E_2)$ is closed.

Since any closed set in $M_1 \times M_2$ is of the form

$$\cap \left[ (F_i^* \times M_2) \cup (M_1 \times F_2^*) \right]$$

where $F_i^*$ is closed in $M_i$, $i = 1, 2$, it follows that $\tilde{I}^{-1}$ is closed and thus that $\tilde{I}$ is continuous.

We are indebted to John Lindberg for suggesting the following example showing that $\tilde{I}^{-1}$ need not be continuous.

Example. Let $A_1$ be the commutative Banach algebra of functions analytic in the interior of the unit disc $D_z$ and continuous on the entire unit disc $D_z$. The maximal ideal space of $A_1$ may be identified with $D_z$ although the $hk$-topology constitutes a genuine weakening of the usual topology of $D_z$. In fact, a $hk$-closed set $F$ in $D_z$ has only countably many points in the (usual) interior of $D_z$ unless $F$ is the whole of $D_z$. Let $A_2$ be $C(D_w)$, i.e., the algebra of all continuous functions on $D_w$. Then the maximal ideal space of $A_2$ in the $hk$-topology is homeomorphic with $D_w$ in its usual topology.

In $A_3$ let $v$ be the element $z \otimes 1 - 1 \otimes w$ ($z$ and $w$ here are the usual complex variables). Then in $M_1 \times M_2$ the set $K = \{(z, w) | z = w\}$ is the hull of $v$, and so $\tilde{I}^{-1}(K)$ is closed in the $hk$-topology of $M_3$. Thus $U = M_3 \setminus \tilde{I}^{-1}(K)$ is open. We shall show that $\tilde{I}(U)$ is not open in $M_1 \times M_2$.

If $\tilde{I}(U)$ were open, there would be open sets $U_i$ in $M_i$ such that $U_1 \times U_2 \subset \tilde{I}(U)$, and $U_1$ could be taken as an ordinary open circle in the interior of $D_w$. Since $F_1 = M_1 \setminus U_1$ is $hk$-closed in $D_w$, $F_1$ has at most countably many points in the ordinary interior of $D_z$. Let $U_2^*$ be the circle in $D_z$ that consists of the same set of complex numbers as the set comprising $U_2$. Then $U_2^* \setminus F_1 \neq \emptyset$ since $F_1$ meets the ordinary interior of $D_z$ in at most countably many points. But then $U_1 \times U_2 \supset (U_2^* \setminus F_1) \times U_2$. Clearly, any point $z$ in $U_2^* \setminus F_1$ corresponds to a point.
NOTE ON THE TENSOR PRODUCT OF BANACH ALGEBRAS

w in $U_2$ such that $z = w$, i.e., $(U_2^* \setminus F_i) \times U_2$ meets $K$ and thus $U_1 \times U_2$ cannot lie in the complement $\hat{K}(U)$ of $K$.

**Remark.** In the last paragraph of p. 302 and the first two lines of p. 303 [2] the proof given is incorrect since the $u_i$ chosen there vary with the $M_i$. The following amended proof should be substituted.

Let $M_{03}$ be in $\mathfrak{M}_3$ and if $\hat{t}(M_{03}) = (M_{01}, M_{02})$, let $N(M_{0i})$ in $\mathfrak{M}_i$ be of the form:

$$N(M_{0i}) = \{M_i\mid |a_{ji}^+(M_i) - a_{ji}^+(M_{0i})| < r_i, j = 1, 2, \ldots, J_i\}, \quad i = 1, 2.$$

To prove the continuity of $\hat{t}$ it suffices to find an $N(M_{03})$ such that $\hat{t}(N(M_{03})) \subseteq N(M_{01}) \times N(M_{02})$. To this end let $u_{0i}$ be identities modulo $M_{0i}, i = 1, 2$ and let

$$N(M_{03}) = \{M_3\mid (a_{j1} \otimes u_{02})^+(M_{03}) - (a_{j1} \otimes u_{02})^+(M_{03}) < s_k, \quad k = 1, 2, j = 1, 2, \ldots, J_1;$$

$$| (u_{01} \otimes a_{j2})^+(M_3) - (u_{01} \otimes a_{j2})^+(M_{03}) | < v_m, \quad m = 1, 2, j = 1, 2, \ldots, J_2;$$

$$| (u_{01} \otimes u_{02})^+(M_3) - 1 | < w \}.$$

We treat two cases: (1) For all $(j, i) a_{ji}^+(M_{0i}) \neq 0$. Then for appropriate choices of the $s_k, v_m$ and $w$, we can assure that

$$\left| (a_{j1} \otimes u_{02})^+(M_{03}) - (a_{j1} \otimes u_{02})^+(M_{03}) \right| = |a_{j1}^+(M_{01}) - a_{j1}^+(M_{03})| < r_1$$

and similarly that $|a_{ji}^+(M_{02}) - a_{ji}^+(M_{03})| < r_2$, where $\hat{t}(M_3) = (M_1, M_2)$. In this case then, $\hat{t}(N(M_{03})) \subseteq N(M_{01}) \times N(M_{02})$. (2) Some $a_{ji}^+(M_{0i}) = 0$. For the $(j, i)$ combinations for which $a_{ji}^+(M_{0i}) \neq 0, s_k, v_m$ and $w$ are chosen as in (1). For the other $(j, i)$ combinations, since $u_{0i}^+(M_{0i})$ is bounded, we see that for suitable $w$ and $M_3$ in $N(M_{03})$, $u_{0i}^+(M_{0i})$ is bounded away from 0 and then for suitable $s_k$ and $v_m, \left| a_{ji}^+(M_{0i}) \right|$ is small. The result follows upon a final revised choice of $s_k, v_m$ and $w$.

**Bibliography**


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