FUNCTIONS OF BVC TYPE[1,2]

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Let f be a measurable function defined on the closed unit square $Q=I\times I$, $I=[0, 1]$. For every $x\in I$, let f_x be the function of y defined by $f_x(y)=f(x, y)$ and for every $y\in I$, let f^y be the function of x defined by $f^y(x)=f(x, y)$. Let $V(f_x)$ and $V(f^y)$ be the variations of f_x and f^y on I, respectively. The function f is said to be BVC (of bounded variation in the sense of Tonelli and Cesari [**1**; **2**]), if there are functions g and h, which are equal to f almost everywhere on Q, such that: $V(g_x)<\infty$ for almost all $x\in I$, $V(h^y)<\infty$ for almost all $y\in I$, $\int_0^1 V(g_x)dx<\infty$ and $\int_0^1 V(h^y)dy<\infty$. The purpose of this note is to show that if f is BVC, then there is a single function k, which is equal to f almost everywhere on Q, such that: $\int_0^1 V(k_x)dx<\infty$ and $\int_0^1 V(k^y)dy<\infty$. This fact has already been established, [**3**], in the special case where f is essentially linearly continuous.

Let f be a function defined on $[a, b]$,

$$P: [a=\beta_0<\beta_1<\cdots<\beta_{r-1}<\beta_r=b]$$

be a partition of $[a, b]$, and define for $x\in(\beta_{m-1}, \beta_m]$, ($x\in[\beta_0, \beta_1]$ if $m=1$), $m=1, 2, \cdots, r$, the functions:

$$\overset{+}{\phi}_P(f; x) = f(a) + \frac{1}{2}\sum_{i=1}^{m}\{[f(\beta_i) - f(\beta_{i-1})] + |f(\beta_i) - f(\beta_{i-1})|\},$$

$$-\overset{-}{\phi}_P(f; x) = \frac{1}{2}\sum_{i=1}^{m}\{[f(\beta_i) - f(\beta_{i-1})] - |f(\beta_i) - f(\beta_{i-1})|\}$$

and if $0\leqq j<k\leqq r$, $v(f; P; \beta_j, \beta_k)=\sum_{i=j+1}^{k}|f(\beta_i)-f(\beta_{i-1})|$. The functions ϕ_P^+, ϕ_P^- are monotone, nondecreasing. The norm of P is defined as $|P|=\max[|\beta_i-\beta_{i-1}|, i=1, 2, \cdots, r]$.

LEMMA. *If f is a BV function on $[a, b]$ and $\{P_n\}$ is a sequence of partitions of $[a, b]$, each a refinement of its predecessor with* $\text{limit}_{n\to\infty}|P_n|=0$, *then* $\text{limit}_{n\to\infty}\phi_{P_n}^+(f; x)$ *and* $\text{limit}_{n\to\infty}\phi_{P_n}^-(f; x)$ *exist at all points of $[a, b]$. If these limits are designated by ϕ^+ and ϕ^- respectively, then $f=\phi^+-\phi^-$ at all points of continuity of f and $V(\phi^+-\phi^-)\leqq V(f)$.*

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PROOF. Let $\{P_n\}$ be a sequence of partitions of $[a, b]$, where each is a refinement of its predecessor and $\text{limit}_{n\to\infty} |P_n| = 0$. Let x be any number in $[a, b]$ and let $(\alpha_n, \beta_n]$ be that subinterval of P_n which contains x, ($[\alpha_n, \beta_n]$ if $x=0$).

Now,

$$\overset{+}{\phi}_{P_n}(f; x) = f(a) + [f(\beta_n) - f(a) + v(f; P_n; a, \beta_n)]/2$$
$$= f(a)/2 + f(\beta_n)/2 + v(f; P_n; a, \alpha_n)/2 + |f(\alpha_n) - f(\beta_n)|/2$$

and

$$\overset{-}{\phi}_{P_n}(f; x) = f(a)/2 - f(\beta_n)/2 + v(f; P_n; a, \alpha_n)/2 + |f(\beta_n) - f(\alpha_n)|/2.$$

But, $\{\alpha_n\}$ is a monotone, nondecreasing sequence, $\{\beta_n\}$ is a monotone, nonincreasing sequence and $\text{limit}_{n\to\infty} \alpha_n = \text{limit}_{n\to\infty} \beta_n = x$. Since f is BV, it follows that $\text{limit}_{n\to\infty} f(\beta_n)$ and $\text{limit}_{n\to\infty} |f(\beta_n) - f(\alpha_n)|$ exist, and $\text{limit}_{n\to\infty} v(f; P_n; a, \alpha_n)$ exists because $\{v(f; P_n; a, \alpha_n)\}$ is a monotone, nondecreasing sequence which is bounded above by the variation of f on $[a, x]$. Thus, $\text{limit}_{n\to\infty} \phi^+_{P_n}(f; x)$ and $\text{limit}_{n\to\infty} \phi^-_{P_n}(f; x)$ exist for all $x \in [a, b]$. Clearly, $\text{limit}_{n\to\infty} [\phi^+_{P_n}(f; x) - \phi^-_{P_n}(f; x)] = \phi^+(f; x) - \phi^-(f; x)$ which equals $f(x+)$ if $x \neq \beta_n$ for any $n = 1, 2, 3, \cdots$, and $f(x)$ if $x = \beta_n$ for some $n = 1, 2, 3, \cdots$, which is just $f(x)$ at all x which are points of continuity of f. In either case, it is clear that $V(\phi^+ - \phi^-) \leqq V(f)$.

THEOREM. *Let f be a measurable function on the square, Q, which is BVC. Then, there is a single function, k, equal to f almost everywhere on Q, for which the sections k_x and k^y are BV almost everywhere in x and y respectively and $\int_0^1 V(k_x)dx < \infty$ and $\int_0^1 V(k^y)dy < \infty$.*

PROOF. Since f is BVC on Q, there are functions g and h, equal to f almost everywhere, such that their sections g_x and h^y are BV almost everywhere in x and y respectively and for which $\int_0^1 V(g_x)dx < \infty$ and $\int_0^1 V(h^y)dy < \infty$.

Let $\{P_n\}$ be a sequence of partitions of I, each one a refinement of the previous one, with the following properties: $\text{limit}_{n\to\infty} |P_n| = 0$; if P_n: $[0 = \beta_0^n \leqq \beta_1^n < \beta_2^n < \cdots < \beta_{r_n}^n \leqq \beta_{r_n+1}^n = 1]$, then $\beta_1^n, \cdots, \beta_{r_n}^n$ are such that $G(x) = g(x, \beta_i^n)$ is summable for $n = 1, 2, 3, \cdots$ and $i = 1, 2, \cdots, r_n$; $\text{limit}_{n\to\infty} \beta_1^n = 0$; $\text{limit}_{n\to\infty} \beta_{r_n}^n = 1$; and β is any element of P_n for all $n = 1, 2, 3, \cdots$ for which $g(x, \beta)$ is summable in x.

For each $n = 1, 2, 3, \cdots$, define, on the interval $[\beta_1^n, \beta_{r_n}^n] \subset I$, functions $\phi^+_{P_n}(g_x, y)$ and $\phi^-_{P_n}(g_x, y)$ exactly as described prior to the lemma where g_x is BV. Then, one defines:

$$\overset{+}{g}_{P_n}(x, y) = \begin{cases} \overset{+}{\phi}_{P_n}(g_x, y) & \text{if } y \in [\beta_1^n, \beta_{r_n}^n] \text{ and } g_x \text{ is BV,} \\ \overset{+}{\phi}_{P_n}(g_x, \beta_1^n) & \text{if } 0 \leqq y \leqq \beta_1^n \text{ and } g_x \text{ is BV,} \\ \overset{+}{\phi}_{P_n}(g_x, \beta_{r_n}^n) & \text{if } \beta_{r_n}^n \leqq y \leqq 1 \text{ and } g_x \text{ is BV,} \\ g(x, y) & \text{if } g_x \text{ is not BV.} \end{cases}$$

Similarly, define $g_{P_n}^-(x, y)$ if g_x is BV and let it be 0 if g_x is not BV.

Consider now, $g_{P_n}^+$ and $g_{P_n}^-$. If $\beta_{j-1}^n < y \leqq \beta_j^n$, $2 \leqq j \leqq r_n$, one has that

$$\overset{+}{g}_{P_n}(x, y) = g_x(\beta_1^n) + \frac{1}{2} \sum_{i=2}^{j} \left\{ [g_x(\beta_i^n) - g_x(\beta_{i-1}^n)] + | g_x(\beta_i^n) - g_x(\beta_{i-1}^n) | \right\}$$

if g_x is BV, i.e. for almost all x. But, since $g(x, \beta_i^n) = g_x(\beta_i^n)$, $i = 1, 2, \cdots, r_n$ is a summable function of x for $n = 1, 2, 3, \cdots$, it follows that $g_{P_n}^+(x, y)$ is a measurable, and in fact summable, function of $(x, y) \in Q$. Similarly $g_{P_n}^-(x, y)$ is a measurable and summable function on Q. Although the form of $g_{P_n}^+$ and $g_{P_n}^-$ is not identical to that of $\phi_{P_n}^+$ and $\phi_{P_n}^-$ in the previous lemma, the only essential distinction is that instead of a $g_x(0)$ term, there is a $g_x(\beta_1^n)$ term appearing, where g_x is BV. Hence, by the lemma, $\text{limit}_{n\to\infty} g_{P_n}^+(x, y)$ and $\text{limit}_{n\to\infty} g_{P_n}^-(x, y)$ exist for all $(x, y) \in Q$. Let g^+ and g^- be these limits, respectively, then g^+ and g^- are measurable since each is a limit of a sequence of measurable functions.

Suppose $0 < \alpha < 1$ and $0 \leqq x \leqq 1$. Then, there is $N > 0$ so that $n > N$ implies P_n is such that $\beta_1^n \leqq \alpha \leqq \beta_{r_n}^n$. Hence, for $n > N$, $g_{P_n}^+(x, \alpha) - g_{P_n}^-(x, \alpha)$ is equal to $\phi_{P_n}^+(g_x, \alpha) - \phi_{P_n}^-(g_x, \alpha)$ if g_x is BV, and $g(x, \alpha)$ if g_x is not BV, and thus, by the lemma, one has that the limit, $g^+(x, \alpha) - g^-(x, \alpha)$, is either $g(x, \alpha)$ or $g(x, \alpha+)$ depending upon whether $\alpha = \beta_j^n$ for some j, $n = 1, 2, 3, \cdots$ and g_x is BV. Thus, $g^+(x, y) - g^-(x, y) = g(x, y)$ at all points $(x, y) \in Q$ such that either g_x is not BV or g_x is BV and continuous at y. Since a BV function can be discontinuous at no more than a countable number of points, if S is the set for which $g^+ - g^-$ differs from g, S is measurable since g^+, g^- and g are measurable, $m(S_x) = 0$ for all x, where $S_x = \{y: (x, y) \in S\}$ and $m_2(S) = \int_0^1 m(S_x)dx$, thus $m_2(S) = 0$. Hence, $g^+ - g^-$ equals g almost everywhere on Q and g_x^+ and g_x^- are monotone for almost all $x \in I$. Also, $V(g_x^+ - g_x^-) \leqq V(g_x)$.

It is clear from the definition of g^+ and g^-, where $\beta \in P_n$, $n=1, 2, 3, \cdots$, that since $|g(x, y)| \leqq |g(x, \beta)| + V(g_x)$, it follows that both $|g^+(x, y)|$ and $|g^-(x, y)|$ are bounded by $|g(x, \beta)| + 2V(g_x)$ and since $V(g_x)$ and $g(x, \beta)$ are both summable on Q, g^+ and g^- are also summable on Q.

Let $(g^+)^s$ and $(g^-)^s$ be the integral means of g^+ and g^-, i.e., $(g^+)^s(x, y) = s^{-2}\int_y^{y+s}\int_x^{x+s} g^+(u, v)dudv$, $0 \leqq x, y \leqq 1$, g^+ is continued periodically and similarly for g^-. It is clear that $(g^+)^s_x$ and $(g^-)^s_x$ are monotone for all x since g_x^+ and g_x^- are monotone for almost all x and it is well known that $(g^+)^s$ and $(g^-)^s$ are continuous and converge almost everywhere on Q to g^+ and g^- as s goes to zero. Thus, $k^+ = \limsup_{s\to 0} (g^+)^s$ and $k^- = \limsup_{s\to 0} (g^-)^s$ have the same properties as g relative to f; i.e., $k = k^+ - k^-$ is equal almost everywhere to f, k_x is BV for almost all x, k_x^+ and k_x^- are monotone and $V(k_x) \leqq V(g_x)$ for almost all x. Thus, $\int_0^1 V(k_x)dx < \infty$.

By exactly the same argument with h, the same function k is obtained due to the symmetry of the integral means with respect to x and y. Thus, there is a single function, k, equal almost everywhere to f, for which k_x and k^y are BV for almost all x and y respectively, and

$$\int_0^1 V(k_x)dx < \infty \quad \text{and} \quad \int_0^1 V(k^y)dy < \infty.$$

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