A PROPERTY OF THE REAL LINE EQUIVALENT TO
THE CONTINUUM HYPOTHESIS

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1. Introduction. Edwin Hewitt has asked if the real line $R$ is a normal topological space (see below for the definition of this term) when it is given the topology $\mathcal{Z}$ which is defined in the following way. Fix a Hamel basis $H$ for $R$ over the rational numbers. For $x \in R$ and $a \in H$, let $x_a$ denote the $a$th coordinate of $x$ in its expansion with respect to $H$. Now, for each countable subset $K \subseteq H$ and each $\epsilon$ such that $0 < \epsilon \leq \infty$, define

$$V(K, \epsilon) = \{x \in R: |x| < \epsilon \text{ and } x_a = 0 \text{ for } a \in K\}.$$ 

Then $\mathcal{Z}$ is the group topology on $R$ that has the collection of all possible $V(K, \epsilon)$ as a basis of open sets at 0. The space $(R, \mathcal{Z})$ is obviously completely regular, since it is a topological group. Theorem 1 answers Hewitt's question. Before stating the theorem, let us recall that a topological space $X$ is normal if any two disjoint closed sets have disjoint neighborhoods.

**Theorem 1.** The real line $R$ under the topology $\mathcal{Z}$ defined above is normal if and only if the continuum hypothesis is true.

2. Proof of sufficiency. First a lemma is stated, next it is pointed out how the lemma implies the desired result, and finally a proof for the lemma is given. For use in stating the lemma, recall that a collection $\mathcal{U}$ of disjoint subsets of a topological space $X$ is discrete if $x \in (\bigcup \mathcal{U})^-$ implies that $x \in U^-$ for some $U \in \mathcal{U}$. (The bar indicates closure in $X$.) As for notation, we adopt the convention that $i$ and $j$ run over all integers $\geq 0$, and $n$ runs over all integers $\geq 1$.

**Lemma 1.** If the continuum hypothesis is true, then there exist a countable number of subsets $R_i$ of $R$ and a countable number of collections $\mathcal{V}_i$ such that

1. $R = \bigcup_i R_i$;
2. for $x \in R_i$, there is a unique $V_{i,z} \in \mathcal{V}_i$ with $x \in V_{i,z}$;
3. for $x \in R_i$, $y \in R_i$, $V_{i,z} \cap V_{i,y} = \emptyset$ if $x \neq y$;
4. for each $i$, $\mathcal{V}_i$ is a discrete open collection.

Let us show how Lemma 1 is applied. From [2, 5.32] it will be seen that it is sufficient to prove that $(R, \mathcal{Z})$ is paracompact. This latter term may be defined as follows. $X$ is paracompact if each open
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cover of \(X\) has an open refinement consisting of a countable union of discrete collections. (See [2, 5.28].) Now, if \(\mathcal{U}\) is an open cover for \(R\), choose for each \(x \in R\) a \(U_x \in \mathcal{U}\) with \(x \in U_x\). Then define \(\mathcal{W}_i = \{ V_{i,x} \cap U_z : x \in R_i \}\). It is easy to verify that \(\mathcal{W} = \bigcup_i \mathcal{W}_i\) has the desired properties.

**Proof of Lemma 1.** In the way of notation, define for each integer \(i \geq 0\)

\[
T_i = \{ x \in R : x_a \neq 0 \text{ for exactly } i \text{ of the } a \in H \}.
\]

It will be proved that there are a countable number of subsets \(T_{i,n}\) of \(T_i\) and a countable number of collections \(\mathcal{W}_{i,n}\) such that

1. \(T_i = \bigcup_n T_{i,n}\);
2. for \(x \in T_{i,n}\), there is a unique \(W_n(x) \in \mathcal{W}_{i,n}\) with \(x \in W_n(x)\);
3. for \(x \in T_{i,n}\) and \(y \in T_{i,n}\), \(W_n(x) \cap W_n(y) = \emptyset\) if \(x \neq y\);
4. for each \(n\), \(W_{i,n}\) is a discrete open collection.

(Note that in (4) we mean discrete and open with respect to \((R, \mathcal{S})\).) It is not difficult to verify that Lemma 1 is a consequence of this.

We begin by using the continuum hypothesis to number the Hamel basis \(H\) from 1 to (but not including) \(\Omega\), where \(\Omega\) is the first uncountable ordinal. For \(1 \leq \alpha < \Omega\), \(a(\alpha)\) will denote the \(\alpha\)th \(a \in H\). Also, for \(x \neq 0\), let \(\delta_x\) be the largest ordinal \(\alpha\) such that \(x_{a(\alpha)} \neq 0\). Now introduce a relation on \(R\) by defining \(y < x\) to mean that \(x \neq y\), but \(y_a = x_a\) only if \(x_a \neq 0\) and \(y_a = 0\). Note that each \(x\) has a finite number of predecessors.

For \(x \neq 0\), define

\[
W(x) = \{ z \in R : z_{a(\alpha)} = x_{a(\alpha)} \text{ for } \alpha \leq \delta_x \};
\]

and for each \(n\), define

\[
W_n(x) = \{ z \in W(x) : |z - y| > 1/n \text{ for all } y < x \}.
\]

When \(i > 0\), let \(\mathcal{W}_{i,n} = \{ W_n(x) : x \in T_i \}\), and set \(T_{i,n} = [\bigcup \mathcal{W}_{i,n}] \cap T_i\).

When \(i = 0\), let \(T_{0,n} = \{ \emptyset \}\) for all \(n\), and let \(\mathcal{W}_{0,n} = \{ R \}\).

Let us check that these sets satisfy (1)–(4) above. This is obvious when \(i = 0\); hence we suppose that \(i > 0\). Clearly, for \(x \in T_i\), one can choose \(n\) so large that \(|x - y| > 1/n\) for all \(y < x\). Hence, for this \(n\), \(x \in W_n(x) \in \mathcal{W}_{i,n}\), and (1) follows from this. Moreover, if \(x\) and \(y\) are both in \(T_i\), \(W(x) \cap W(y) = \emptyset\) only if \(x = y\); hence, for \(x\) and \(y\) in \(T_{i,n}\), \(W_n(x) \cap W_n(y) = \emptyset\) only if \(x = y\). Property (3) is a consequence of this, and (2) follows from this and the definition of \(T_{i,n}\).

In proving (4), let us first verify that \(\mathcal{W}_{i,n}\) is an open collection. Note that each \(W(x)\) is open, since \(W(x) = x + V(K, \infty)\), where \(K = \{ a(\alpha) : \alpha \leq \delta_x \}\). Also, since each member of the usual topology for \(R\) is in \(\mathcal{S}\), \(U = \{ z \in R : |z - y| > 1/n \text{ for all } y < x \}\) is in \(\mathcal{S}\). Hence \(W_n(x) = W(x) \cap U\) is open.
Finally, to show that \( W_{i,n} \) is a discrete collection, suppose \( z \in (U \cup W_{i,n})^{-} \). Now \( z \in T_w \) for some integer \( p \). If \( p \geq i \), then let \( y \) be defined by choosing \( y_a = z_a \) except for the \( p - i \) largest \( a \) for which \( z_a \neq 0 \) (where "largest" refers to the ordering of \( H \)); for the excepted \( a \), let \( y_a = 0 \). For this \( y \), \( z \in W(y) \) and \( y \in T_{i,n} \); hence \( W(y) \) is a neighborhood of \( z \) which meets only one member of \( W_{i,n} \), namely \( W_n(y) \).

All that remains is the case \( p < i \). It will be proved that, for \( z \) and \( p \) as above, this case cannot occur. Suppose \( z \in T_p, p < i \), and let

\[
U = \{ y \in W(z) : |y - z| < 1/n \}.
\]

We will prove that \( U \) is a neighborhood of \( z \) which does not meet \( U \cup W_{i,n} \). Suppose to the contrary that \( U \cap W_n(x) = \emptyset \) for some \( x \in T_i \), and choose a \( y \in U \cap W_n(x) \). Note that \( z < x \), since \( U \cap W_n(x) = \emptyset \) implies that \( W(z) \cap W(x) = \emptyset \), and the latter implies that \( x_{\alpha(a)} = z_{\alpha(a)} \) for all \( \alpha \leq \delta_a \). This leads to a contradiction because \( y \in W_n(x) \) and \( z < x \) imply \( |y - z| > 1/n \); but \( y \in U \) implies \( |y - z| < 1/n \). (Note that \( U \) is open for the same reason that each \( W_n(x) \) is open.)

**Corollary.** Assume the continuum hypothesis. Then every subspace of \((R, 3)\) is paracompact, and hence normal. Also, every subset of \((R, 3)\) is an \( F_\sigma \). (An \( F_\sigma \) is a subset which is the union of a countable number of closed subsets.)

**Proof.** Note that Lemma 1 remains true if \( R \) is replaced by a subspace \( A \) of \( R \). Hence \( A \) is paracompact as above. Also, observe that the properties of \( W_i \) imply that each subset of \( R_i \) is closed. Hence \( A = U_i (R_i \cap A) \) with each \( R_i \cap A \) closed in \((R, 3)\).

**Remark.** The last assertion of the corollary can be derived without the continuum hypothesis by an argument similar to that appearing in the first paragraph of the proof of Lemma 2 in the next section.

**3. Proof of necessity.** Throughout this section it will be assumed that \( \aleph_1 < 2^{\aleph_0} \). We will suppose that \((R, 3)\) is normal and argue for a contradiction. This will follow a sequence of lemmas.

Let \( i \) be the least integer such that \( T_i \) is of second category in \( R \)—where, from here on, \( R \) will denote the real numbers under their usual topology.

**Lemma 2.** \( X = \bigcup_j T_{i+j} \) is a normal subspace of \((R, 3)\).

**Proof.** It will be proved that \( X \) is an \( F_\sigma \) in \((R, 3)\). It is sufficient to show this, since it is well known that an \( F_\sigma \) in a normal space is normal. Define, for integers \( j \) and \( n \), \( T_{j,n} = \{ x \in T_j : |x - y| > 1/n \text{ for all } y < x \} \). (The \( T_{j,n} \) of the last section could have been defined this way.) One may easily verify that \( T_j = U_n T_{j,n} \). Hence it will suffice to prove that each \( T_{j,n} \) is closed in \((R, 3)\).
To accomplish the latter, first suppose that \( y \in T_k \setminus T_{j,n} \) (\( k \geq j \)) and \( K = \{ a \in H : y_a \neq 0 \} \). Then \( y + V(K, \infty) \) is an open neighborhood of \( y \) which does not meet \( T_{j,n} \). Now assume that \( y \in T_k \) (\( k < j \)) and that \( K \) is as above. Then \( y + V(K, 1/n) = V \) does not meet \( T_{j,n} \). In fact, if \( x \in T_j \cap V \) then \( y_b = x_b \) for all \( b \) such that \( y_b \neq 0 \). Hence \( y < x \), but \( |x - y| < 1/n \). It follows that \( x \in T_{j,n} \).

For convenience, let us suppose that, whenever a rational number is written in the form \( m/n \), we have \( m \) and \( n \) relatively prime. Define \( A \) to be the set of \( x \in T_i \) such that, for any \( a \), the \( m \) in the expression \( x_a = m/n \) is even. Let \( B = T_i \setminus A \). It follows by an argument similar to part of the proof of Lemma 2 that each subset of \( T_i \) is closed in \( X \). Hence \( A \) and \( B \) are both closed in \( X \). By the normality of \( X \), one may choose disjoint \( U \) and \( V \), each open subsets of \( X \), such that \( A \subset U \) and \( B \subset V \).

By the definition of the topology \( \mathfrak{S} \), one may choose, for each \( x \in A \cup B \), a countable \( K(x) \subset H \) and \( \varepsilon(x) > 0 \) such that \( V(x) = [x + V(K(x), \varepsilon(x))] \cap X \) is contained in \( U \) if \( x \in A \), or in \( V \) if \( x \in B \). Using this notation we will now describe some of the properties of \( A \) and \( B \).

**Lemma 3.** \( A \) is of second category in \( R \).

**Proof.** Either \( A \) or \( B \) is of second category in \( R \), since \( A \cup B = T_i \). For \( x \in B \), \( 2^nx \in A \) if \( n \) is large. Let \( B_n = \{ x \in B : 2^nx \in A \} \). If \( B \) is of second category in \( R \), then so is \( B_n \) for some \( n_0 \). Hence \( A \), which contains \( 2^nx \), is of second category in \( R \).

For \( S \subset R \) and \( K \subset H \), let \( S(K) = \{ x \in S : x_a = 0 \text{ when } a \in K \} \).

**Lemma 4.** If \( S \subset T_i \) is of second category in \( R \), then \( S(K) \) is of second category in \( R \) for each countable subset \( K \) of \( H \).

**Proof.** Let \( gK \) denote the group generated by \( K \). Suppose \( x \in gK \setminus \{ 0 \} \). Let \( S_x = (S-x)(K) \). Since \( S_x \subset \{ T_r : 0 \leq r < i \} \), \( S_x \) is of first category in \( R \) by the minimality of \( i \). Hence \( S_x + x \) is of first category in \( R \). Since \( K \) is countable, \( T = \bigcup \{ S_x + x : x \in gK \setminus \{ 0 \} \} \) is of first category in \( R \). Consequently, \( S(K) = S \setminus T \) is of second category in \( R \).

For each set \( S \) let \( |S| \) denote the cardinality of \( S \). Let us say that an interval \( I \) (all intervals which occur here will be open) is \( \aleph_2 \)-filled with a subset \( S \) of \( R \) if, for each \( K \subset H \), \( |K| < \aleph_2 \), one has \( |S(K) \cap I| \geq \aleph_2 \).

**Lemma 5.** If \( I \) is \( \aleph_2 \)-filled with a set \( S \) which is the union of a countable collection of \( S_n \), then for some \( n_0 \) there are arbitrarily small intervals contained in \( I \) which are \( \aleph_2 \)-filled with \( S_{n_0} \).
PROOF. Suppose to the contrary that, for each \( n \), there is some \( \theta(n) > 0 \) such that, given an interval \( J \) of length less than \( \theta(n) \), one may choose \( K_J \subset H \) with the property that \([ J \cap S_n(K_J) ] < \aleph_1 \) and \( | K_J | < \aleph_2 \). For a fixed \( n \), let \( J_1, J_2, \ldots \) be a countable cover of \( I \) by intervals of length less than \( \theta(n) \). Define \( K_n \) as \( \bigcup_j S_n(K_J) \), and \( K \) as \( \bigcup_n K_n \). Then \( S_n(K_n) \subset \bigcap_j S_n(K_J) \); hence \( S_n(K_n) \cap I \subset \bigcup_j (S_n(K_J) \cap J_j) \), and \( | S_n(K_n) \cap I | < \aleph_2 \). Also, \( S(K) \subset \bigcup_n S_n(K_n) \); hence \( | S(K) \cap I | \leq | \bigcup_n (S_n(K_n) \cap I) | < \aleph_2 \). Since \( | K | < \aleph_2 \), this contradicts the assumption that \( I \) is \( \aleph_2 \)-filled with \( S \).

**Lemma 6.** Each interval \( I \) is \( \aleph_2 \)-filled with \( B \).

**Proof.** Suppose \( | K | < \aleph_2 \). Then \( | B(K) | > \aleph_1 \) as one may easily verify. Let \( x \in R \setminus \{ 0 \} \) be chosen such that each interval containing \( x \) also contains \( \aleph_2 \) elements of \( B(K) \). Choose a rational number \( r \) of the form \( m/2^n \) \((n > 0) \) such that \( rx \in I \). Note that \( rB(K) \subset B(K) \), since \( m \) and \( 2^n \) are relatively prime. Hence \( | B(K) \cap I | \geq | rB(K) \cap I | \geq \aleph_2 \).

For the remainder of this section, let \( n \) be an integer, and let \( I \) be an interval, such that \( n \) and \( I \) have the properties of the next lemma. (For the statement of Lemma 7, recall the definition of \( \varepsilon(x) \) in the expression of \( V(x) \).)

**Lemma 7.** There are an integer \( n \) and an interval \( I \) such that (1) if \( A_n = \{ x \in A : \varepsilon(x) > 1/n \} \) then \( A_n \cap I \) is of second category in \( R \), (2) \( I \) is \( \aleph_2 \)-filled with \( B_n \) where \( B_n = \{ x \in B : \varepsilon(x) > 1/n \} \), and (3) \( I \) is of length less than \( 1/n \).

**Proof.** For (1) choose an integer \( p \) such that \( A_p \) is of second category in \( R \). It is not difficult to show that, for some interval \( J, A_p \cap L \) if of second category in \( R \) for each subinterval \( L \) of \( J \). Now \( J \) is \( \aleph_2 \)-filled with \( B \) by Lemma 6, and by Lemma 5 there is a \( q \) such that there are arbitrarily small subintervals of \( J \) which are \( \aleph_2 \)-filled with \( B_q \). Let \( n = \max(p, q) \). Choose \( I \) to be a subinterval of \( J \) such that (3) is satisfied for \( I \) and (2) is satisfied for \( B_n \) and \( I \). It follows that (1), (2), and (3) are satisfied for \( n \) and \( I \).

This completes the sequence of lemmas, and we now observe the following facts:

(i) If \( x \in (A_n \cup B_n) \cap I \), then \( V(x) \cap I = W(x) \cap I \), where \( W(x) = \text{Reg}(x, \infty) \). For suppose \( y \in W(x) \cap I \); then \( | y - x | < 1/n < \varepsilon(x) \), and \( y \in V(x) \cap I \).

(ii) If \( y \in W(x) \cap W(x') \) for \( x \in A_n \cap I \) and \( x' \in B_n \cap I \), then there is some \( y' \in W(x) \cap W(x') \cap I \). In fact, let \( a \in H \setminus (K(x) \cup K(x')) \) be chosen such that \( y_a = 0 \). Now pick a rational number \( r \) such that \( y' = y + ra \in I \). It follows that \( y' \) has the desired property.
(iii) For each \( x \in A_n \cap I \) and each \( x' \in B_n \cap I \), \( W(x) \cap W(x') = \emptyset \). In fact, suppose \( y \in W(x) \cap W(x') \) for some choice of \( x \in A_n \cap I \) and \( x' \in B_n \cap I \). Pick \( y' \in X \) as in (ii). Then by (i), \( y' \in V(x) \cap V(x') \), which contradicts the assumption that \( U \cap V = \emptyset \).

This section will be completed by proving that (iii) is false. To accomplish this, choose by induction on \( \alpha \) an
\[
x(\alpha) \in (I \cap A_n) (\bigcup \{ K(x(\beta)) : \beta < \alpha \})
\]
for \( 1 \leq \alpha \leq \Omega \) such that
\[
(x(\alpha))_a = 0 \text{ if } (x(\beta))_a \neq 0 \text{ for any } \beta < \alpha.
\]
This is accomplished by using Lemma 4 and the fact that \( A_n \cap I \) is of second category in \( R \). Now pick \( x_0 \in B_n \cap I \) such that \( (x_0)_a = 0 \) if either \( a \in \bigcup \{ K(x(\alpha)) : 1 \leq \alpha < \Omega \} \) or \( (x(\alpha))_a \neq 0 \) for some \( 1 \leq \alpha < \Omega \).

We will contradict (iii) by proving that, for a sufficiently large \( \alpha \), \( W(x_0) \cap W(x(\alpha)) \neq \emptyset \). In fact, pick \( \alpha \) large enough so that \( (x(\alpha))_a = 0 \) if either \( a \in K(x_0) \) or \( (x_0)_a \neq 0 \). (This is possible because of condition (i).) One may verify that, for this \( \alpha \), \( x_0 + x(\alpha) \in W(x_0) \cap W(x(\alpha)) \).

This completes the proof.

4. Remarks. (a) The first three lemmas of §3 can be either much simplified or omitted when the Hamel basis is of second category in \( R \). However, this is not always so; for instance, a maximal linearly independent subset \( H \) of a set which is both of positive measure and of first category in \( R \) can be proved to be a Hamel basis. On the other hand, V. L. Klee has pointed out to me that a method due to F. B. Jones [Bull. Amer. Math. Soc. vol. 48 (1942) pp. 115-120] demonstrates the existence of a Hamel basis \( H' \) which is of second category in \( R \).

As in §1, define a topology \( \mathfrak{S} \) on \( R \) using \( H \), and a topology \( \mathfrak{S}' \) using \( H' \), where \( H \) is of first category in \( R \) and \( H' \) is of second category in \( R \). One might reasonably conjecture that \( (R, \mathfrak{S}) \) and \( (R, \mathfrak{S}') \) are homeomorphic. However, let \( i \) be the identity function on \( R \), and let \( f \) be defined by extending linearly a fixed one-to-one correspondence between \( H \) and \( H' \). It can be proved that \( f \) is not continuous and that \( i \) is not necessarily continuous, where both are considered as functions from \( (R, \mathfrak{S}) \) to \( (R, \mathfrak{S}') \). Hence, it does not seem that a natural homeomorphism exists, and it is necessary to have a proof that applies to either possibility for \( H \).

(b) There are several other topologies for \( R \) whose properties complement those of \( \mathfrak{S} \). Define \( \mathfrak{S}(N, \epsilon) \) to be the group topology on \( R \).
generated by all $V(K, \epsilon)$ with $|K| < \aleph$ and $\epsilon > 0$, and define $3(\aleph)$ as the group topology generated by all $V(K, \infty)$ with $|K| < \aleph$. Without any assumptions about the continuum hypothesis one may prove that $3(\aleph_0, \epsilon)$ is not normal, that $3(2^{\aleph_0}, \epsilon)$ is normal (and paracompact), and that $3(\aleph)$ is normal (and paracompact) for any $\aleph$. Moreover, for $3(\aleph)$ with $\aleph \leq \aleph_1$, it may be proved that this topology is Lindelöf by the methods of [1, Proposition 3]. (A space is Lindelöf if each open cover has a countable subcover. See [2, 5.1] for a proof that Lindelöf implies paracompact.)

(c) Note that it is also true that $2^{\aleph_0} = \aleph_1$ if and only if $(\mathbb{R}, 3)$ is paracompact; but a proof of this may be constructed which is not so involved as that for Theorem 1.

REFERENCES


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