

CONTIGUOUS SPHERE CLUSTERS. I. ISOMORPHIC CHAINS

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One of the remarkable phenomena (still unexplained) concomitant with the transmission of the "code of heredity" to succeeding generations (in sexually reproducing organisms) is "pairing" of the chromosomes at zygote: That is, during the zygotene stage of meiosis (the process whereby the sex cells are formed) homologous chromosomes become "contiguous" at homologous sites (the chromomeres) throughout their length.

In this paper we consider an analogous geometric situation using "pairing at zygote" as our prototype. Our "geometric chromosome" will be a "chain" of (not necessarily equal) spheres. Now, the fact that two "isomorphic" sphere chains (i.e. "twins") can achieve (simultaneous) contiguity between corresponding spheres, has implications for the structure of the chains. Thus, e.g., what about the relative sizes of the constituent spheres? It is the purpose of the present study to derive some of these structural implications.

1. Definitions and statement of the theorem. Let s_i and c_i denote the radius and center of the 3-sphere S_i , respectively. Let $l(c_i, c_j)$ denote the line passing through the points c_i and c_j .

By a *twin-chain* $[S_1, \dots, S_n; S'_1, \dots, S'_n]$ of spheres we shall mean a configuration of (Euclidean) 3-spheres

$$(1.1) \quad S_1, S_2, \dots, S_n, S'_1, S'_2, \dots, S'_n$$

such that (i) no two spheres of (1.1) have interior points in common, (ii) $s_i = s'_i$ ($i = 1, \dots, n$) (s'_i is the radius of S'_i), and (iii) S_i is tangent to S_{i+1} ($i = 1, \dots, n - 1$), S'_i is tangent to S'_{i+1} ($i = 1, \dots, n - 1$), and S_i is tangent to S'_i ($i = 1, \dots, n$).

Twin-chains $[S_1, \dots, S_n; S'_1, \dots, S'_n]$ and $[T_1, \dots, T_n; T'_1, \dots, T'_n]$ will be referred to as *equivalent* if $s_i = t_i$ ($i = 1, \dots, n$) (t_i is the radius of T_i).

If spheres (1.1) are "moved" (possibly with varying mutual relative positions) so that at each instant t of a time interval $0 \leq t \leq 1$, properties (i), (ii), (iii) (in the definition of twin-chain) hold, then we shall say that the twin-chain $K(0): [S_1, \dots, S_n; S'_1, \dots, S'_n]_0$ at $t=0$ has been *deformed* into the twin-chain $K(1): [S_1, \dots, S_n; S'_1, \dots, S'_n]_1$ at $t=1$; we shall refer to $K(1)$ as a *deform* of $K(0)$.

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$K(0)$ and $K(1)$ are, of course, equivalent. The set of all possible deforms of $K(0)$ will be referred to as a *deformation class*. Now, let S be the set of all twin-chains which are equivalent to a given twin-chain K . Then S decomposes into deformation classes such that if $X, Y \in S$, then X and Y belong to the same deformation class if and only if X is a deform of Y .

We shall refer to " n " (the highest subscript appearing in the bracket symbol for a twin-chain) as the *length* of the chain.

Now, all equivalent twin-chains of length two belong to the same deformation class. However, not all equivalent twin-chains of length three necessarily belong to the same deformation class. Consider the following example: On a Cartesian coordinate system in E_3 let S_1, S_2, S'_1, S'_2 be defined as follows:

$$S_1: (x+1)^2 + y^2 + (z-1)^2 = 1, \quad S'_1: (x-1)^2 + y^2 + (z-1)^2 = 1,$$

$$S_2: (x+1)^2 + y^2 + (z+1)^2 = 1, \quad S'_2: (x-1)^2 + y^2 + (z+1)^2 = 1.$$

Let

$$T_3: x^2 + (y-R)^2 + z^2 = R^2, \quad T'_3: x^2 + (y+R)^2 + z^2 = R^2$$

be spheres each tangent to S_1, S_2, S'_1, S'_2 , and

$$U_3: x^2 + (y-R)^2 + (z-\alpha)^2 = R^2,$$

$$U'_3: x^2 + (y+R)^2 + (z-\alpha)^2 = R^2$$

($\alpha < 0$) be spheres each tangent to S_2 and S'_2 . Since

$$(1.2) \quad [S_1, S_2, T_3; S'_1, S'_2, T'_3]$$

permits no deformation (except those deforms which belong to the same congruence class) and

$$(1.3) \quad [S_1, S_2, U_3; S'_1, S'_2, U'_3]$$

does, (1.2) and (1.3) belong to distinct deformation classes.

A twin-chain $[S_1, S_2, S_3; S'_1, S'_2, S'_3]$ such that

(i) the points P_i of contact between S_i and S'_i ($i=1, 2, 3$) are collinear on a line L ,

(ii) $l(c_i, c'_i)$ (c'_i the center of S'_i) is perpendicular to L ($i=1, 2, 3$),

(iii) $l(c_1, c'_1)$ is perpendicular to $l(c_3, c'_3)$,

will be called *normal*. We shall refer to a normal twin-chain $[S_1, S_2, S_3; S'_1, S'_2, S'_3]$ as being of *types* (1, 2, 3) or (1, 3, 2) accordingly as P_2 falls between P_1 and P_3 , or P_3 falls between P_1 and P_2 on L .

We prove the following theorem:

THEOREM. (i) If $[S_1, S_2, S_3; S'_1, S'_2, S'_3]$ is normal of type (1, 2, 3), then

$$(1.4) \quad s_2 \geq \frac{s_1 s_3 (s_1 + s_3 - \{2s_1 s_3\}^{1/2})}{2(s_1^2 + s_3^2)}.$$

(ii) If s_1, s_2, s_3 are positive numbers such that (1.4) holds, then it is possible to construct normal twin-chains $[S_1, S_2, S_3; S'_1, S'_2, S'_3]$ of type (1, 2, 3), where s_1, s_2, s_3 are the radii of S_1, S_2, S_3 respectively.

COROLLARY 1. If $[S_1, S_2, S_3; S'_1, S'_2, S'_3]$ is normal of type (1, 2, 3), then $s_2 \geq \{(2 - 2^{1/2})/4\} \cdot \text{Min}(s_1, s_3)$.

From the manner in which a twin-chain can be deformed, it may be possible to obtain certain estimates of the relative sizes of its constituent parts:

COROLLARY 2. If $[S_1, \dots, S_n; S'_1, \dots, S'_n]$ can be deformed so that for some i ($1 < i < n$) $[S_{i-1}, S_i, S_{i+1}; S'_{i-1}, S'_i, S'_{i+1}]$ becomes normal, then (1.4) holds with $i-1, i, i+1$ replacing the subscripts 1, 2, 3 respectively.

2. PROOF OF (i). Let $A, B, C, A', B', C', a, b, c$ denote $S_1, S_2, S_3, S'_1, S'_2, S'_3, s_1, s_2, s_3$ respectively. a, b, c are positive by hypothesis; there is no loss in generality if we assume $a \geq c$ (since in the contrary case we can reletter the spheres).

Let a three-dimensional cartesian coordinate system be superimposed on $[A, B, C; A', B', C']$ (which is normal, of type (1, 2, 3)) so that the centers of A, B, C, A', B', C' are

$$(2.1) \quad (-a, 0, 0), (-l, -m, n), (0, -c, u) \quad (a, 0, 0), (l, m, n), (0, c, u)$$

respectively, with

$$(2.2) \quad u > n > 0.$$

This uniquely determines the orientation (i.e. the right or left handedness) of the system.

Now, $l^2 + m^2 = b^2$. Since A' and B' are tangent, and B' and C' are tangent, we have

$$(l - a)^2 + m^2 + n^2 = (a + b)^2$$

and

$$l^2 + (m - c)^2 + (u - n)^2 = (b + c)^2$$

and thus

$$(2.3) \quad n = \{2a(b+l)\}^{1/2}$$

and

$$(2.4) \quad u = n + \{2c(b+m)\}^{1/2}.$$

Since A' and C' have no interior points in common, we have $a^2 + c^2 + u^2 \geq (a+c)^2$, which implies

$$(2.5) \quad u \geq (2ac)^{1/2}.$$

Combining (2.3), (2.4) and (2.5), we obtain

$$\left(\frac{b+m}{a}\right)^{1/2} + \left(\frac{b+l}{c}\right)^{1/2} \geq 1.$$

Also, since A and B' have no interior points in common, and C and B' have no interior points in common, we have

$$(l+a)^2 + m^2 + n^2 \geq (a+b)^2$$

and

$$l^2 + (m+c)^2 + (n-u)^2 \geq (b+c)^2$$

which imply $l \geq 0$ and $m \geq 0$, respectively.

Thus, the normalcy of $[A, B, C; A', B', C']$ implies

$$(2.6) \quad \begin{aligned} \left(\frac{b+m}{a}\right)^{1/2} + \left(\frac{b+l}{c}\right)^{1/2} &\geq 1, \\ m^2 + l^2 &= b^2, \\ l \geq 0, \quad m &\geq 0. \end{aligned}$$

We shall prove (i) of the theorem, by showing that (2.6) implies

$$(2.7) \quad b \geq \frac{ac(a+c - \{2ac\}^{1/2})}{2(a^2 + c^2)}.$$

If in (2.6) we let

$$(2.8) \quad x = \frac{b+m}{a}, \quad y = \frac{b+l}{c},$$

we obtain

$$(2.9) \quad \begin{aligned} x^{1/2} + y^{1/2} &\geq 1, \\ \frac{(x-b/a)^2}{(b/a)^2} + \frac{(y-b/c)^2}{(b/c)^2} &= 1, \\ x &\geq b/a, \quad y \geq b/c. \end{aligned}$$

We now note that if x_0, y_0, b_0 satisfies (2.9), then

$$(2.10) \quad m_0 \equiv ax_0 - b_0, \quad l_0 \equiv cy_0 - b_0, \quad b_0$$

satisfies (2.6). Also, if m_0, l_0, b_0 satisfies (2.6), then

$$(2.11) \quad x_0 \equiv \frac{b_0 + m_0}{a}, \quad y_0 \equiv \frac{b_0 + l_0}{c}, \quad b_0$$

satisfies (2.9). Thus if we find a triplet (x, y, b) satisfying (2.9) with minimal positive b , we obtain through (2.10) a triplet (m, l, b) satisfying (2.6) with minimal positive b .

On a two-dimensional cartesian coordinate system, let R denote the set of all points for which $x^{1/2} + y^{1/2} \geq 1$.

Let E_b denote the second expression of (2.9); E_b is a single parameter family of ellipses, with b the parameter. Each member of E_b has its center at $(b/a, b/c)$ with axes of symmetry parallel to the x - and y -axes, and is tangent to the x - and y -axes at the points $(b/a, 0)$ and $(0, b/c)$ respectively. Since $a \geq c$, the line $y = ax/c$ on which the centers of the members of E_b lie, has slope greater or equal to one for any (positive) value of b . Each member of E_b is "upright", i.e. its major axis is parallel to the y -axis. Now, if (x, y) is a point on some member E_{b_i} of E_b such that $x \geq b/a, y \geq b/c$ and $(x, y) \in R$, then (x, y, b_i) will satisfy (2.9).

Now, let E_{b_m} be that member of E_b which (i) has no interior point in common with R , and (ii) is tangent to $x^{1/2} + y^{1/2} = 1$ in a point P . It is clear that the abscissa and ordinate of P are greater than or equal to b_m/a and b_m/c respectively ($(b_m/a, b_m/c)$ is the center of E_{b_m}). Since E_{b_n} , for $b_n < b_m$, has no point in common with R , then b_m is the minimum value of b which may be the third member of a triplet (x, y, b) satisfying (2.9).

Now, if $x^{1/2} + y^{1/2} = 1$ and a member of E_b are tangent at a point (x, y) , then (x, y) must satisfy both equations, and the slopes of each at (x, y) must be equal, i.e. (x, y) must satisfy the system

$$(2.12) \quad \begin{aligned} x^{1/2} + y^{1/2} &= 1, \\ \frac{(x - b/a)^2}{(b/a)^2} + \frac{(y - b/c)^2}{(b/c)^2} &= 1, \\ c^2(y - b/c)y^{1/2} &= a^2(x - b/a)x^{1/2}. \end{aligned}$$

Thus, our problem becomes that of finding the smallest value of b for which (2.12) has a solution (x, y) .

Now, if we let

$$(2.13) \quad x^{1/2} = f, \quad y^{1/2} = g,$$

then (2.12) becomes

$$(2.14) \quad \begin{aligned} f + g &= 1, \\ \frac{(f^2 - b/a)^2}{(b/a)^2} + \frac{(g^2 - b/c)^2}{(b/c)^2} &= 1, \\ c^2(g^2 - b/c)g &= a^2(f^2 - b/a)f. \end{aligned}$$

We note that any solution (f, g, b) of (2.14) indicates (through (2.13)) a solution (x, y, b) of (2.12) (since only positive x, y need be considered), and conversely. Therefore a triplet (f, g, b) satisfying (2.14) with minimal b provides a triplet (x, y, b) satisfying (2.12) with minimal b .

Now, replacing f by $1-g$ in the second and third equations of (2.14), and simplifying, we obtain

$$(2.15) \quad b^2 + Gb + H = 0,$$

where

$$\begin{aligned} G &= -2(a+c)g^2 + 4ag - 2a, \\ H &= (a^2 + c^2)g^4 - 4a^2g^3 + 6a^2g^2 - 4a^2g + a^2, \end{aligned}$$

and

$$(2.16) \quad Mb + N = 0$$

where

$$M = -(a+c)g + a, \quad N = (a^2 + c^2)g^3 - 3a^2g^2 + 3a^2g - a^2.$$

The eliminant between (2.15) and (2.16) is

$$(2.17) \quad \begin{vmatrix} 1 & G & H \\ 0 & M & N \\ M & N & 0 \end{vmatrix} = 0$$

which becomes

$$(2.18) \quad acg^2(g-1)^2\{2(a^2+c^2)g^2 - 4a^2g + 2a^2 - ac\} = 0.$$

We note that $g=0, g=1$ are the solutions of (2.18) which correspond to the tangency between $x^{1/2}+y^{1/2}=1$, and members E_{b_0}, E_{b_1} of E_b at the points $(1, 0)$ and $(0, 1)$ respectively; b_0 and b_1 cannot be minimal since E_{b_0} and E_{b_1} have "internal" points in common with R , respectively. Thus, setting the expression in the braces of (2.18) equal to zero, and solving, we obtain

$$(2.19) \quad g = \frac{2a^2 \pm (a-c)(2ac)^{1/2}}{2(a^2 + c^2)}.$$

Solving (2.15) and (2.16) for b , we obtain

$$(2.20) \quad b = cg^2 + a(1-g)^2 \pm (2ac)^{1/2}(1-g)g$$

and

$$(2.21) \quad b = \frac{c^2g^3 + a^2(g-1)^3}{cg + a(g-1)},$$

respectively. Thus, for the two values of g given by (2.19) we obtain two values for b given by (2.21), and four more values given by (2.20). Of these six values of b we shall determine the smallest.

Let $b(g)$ denote the function of g which is on the right side of (2.21). Let g_+ denote the expression on the right side of (2.19) taken with the plus sign, and g_- the same expression taken with the minus sign. We show that

$$(2.22) \quad b(g_+) \geq b(g_-).$$

If $a=c$, then (2.22) obviously holds since $g_+ = g_-$. Suppose $a > c$. Writing V for $(a-c)(2ac)^{1/2}$ we have

$$\begin{aligned} b(g_+) &= \frac{1}{4(a^2 + c^2)} \left(\frac{V^3 + 12a^2c^2V + 8a^2c^2(a^2 - c^2)}{(a+c)V + 2ac(a-c)} \right) \\ &= \frac{ac}{2(a^2 + c^2)} \left(\frac{(a+c)^2 + 2(a+c)(2ac)^{1/2} + 2ac}{(a+c) + (2ac)^{1/2}} \right) \\ &= \frac{ac(a+c + \{2ac\}^{1/2})}{2(a^2 + c^2)}. \end{aligned}$$

In a similar manner we can show that

$$(2.23) \quad b(g_-) = \frac{ac(a+c - \{2ac\}^{1/2})}{2(a^2 + c^2)}.$$

Comparing $b(g_+)$ and $b(g_-)$, we see that (2.22) holds.

We now note that since $a, c > 0$, and $a \geq c$, the inequalities

$$\begin{aligned} cg_-^2 + a(1-g_-)^2 + (2ac)^{1/2}(1-g_-)g_- \\ \geq cg_-^2 + a(1-g_-)^2 - (2ac)^{1/2}(1-g_-)g_-, \\ cg_+^2 + a(1-g_+)^2 + (2ac)^{1/2}(1-g_+)g_+ \\ \geq cg_-^2 + a(1-g_-)^2 - (2ac)^{1/2}(1-g_-)g_-, \end{aligned}$$

$$cg_+^2 + a(1 - g_+)^2 - (2ac)^{1/2}(1 - g_+)g_+ \geq cg_-^2 + a(1 - g_-)^2 - (2ac)^{1/2}(1 - g_-)g_-$$

hold. Thus

$$(2.24) \quad cg_-^2 + a(1 - g_-)^2 - (2ac)^{1/2}(1 - g_-)g_-$$

is the least of the four values of b obtained from (2.19) and (2.20). Furthermore (2.24) is positive since

$$cg_-^2 + a(1 - g_-)^2 \geq 2(ac)^{1/2}(1 - g_-)g_- > (2ac)^{1/2}(1 - g_-)g_-.$$

Finally we note that (2.23) is identical to (2.24). Writing K for $2a^2 - (2ac)^{1/2}(a - c)$, and L for $2c^2 + (2ac)^{1/2}(a - c)$, we have

$$\begin{aligned} cK^2 + aL^2 - (2ac)^{1/2}KL &= 2a^3c^2 + 2a^2c^3 + 2ac^4 + 2a^4c - 2a^3c(2ac)^{1/2} - 2ac^3(2ac)^{1/2} \\ &= ac(a + c - \{2ac\}^{1/2})(K + L); \end{aligned}$$

dividing the first and last expressions by $(K + L)^2$ we see that (2.23) is identical to (2.24). Thus (2.23) is the smallest of the six b 's. This completes the proof of part (i) of the theorem.

PROOF OF COROLLARY 1. From (1.4) we obtain

$$\begin{aligned} s_2 &\geq \frac{s_1 + s_3 - \{2s_1s_3\}^{1/2}}{2\left(\frac{s_1}{s_3} + \frac{s_3}{s_1}\right)} \geq \frac{s_1 + s_3 - \{2s_1s_3\}^{1/2}}{2\left(\frac{\max(s_1, s_3)}{\min(s_1, s_3)} + 1\right)} \\ &= \left(\frac{s_1 + s_3 - \{2s_1s_3\}^{1/2}}{2(s_1 + s_3)}\right) \min(s_1, s_3) = \left(1 - \frac{\{2s_1s_3\}^{1/2}}{s_1 + s_3}\right) \frac{\min(s_1, s_3)}{2}. \end{aligned}$$

Since $s_1 + s_3 \geq 2\{s_1s_3\}^{1/2}$, we have $s_2 \geq \{(2 - 2^{1/2})/4\} \cdot \text{Min}(s_1, s_3)$.

3. PROOF OF (ii). Let $a, b, c > 0$ with $a \geq c$ satisfy

$$(3.1) \quad b \geq \frac{ac(a + c - \{2ac\}^{1/2})}{2(a^2 + c^2)}.$$

We show how to construct a normal twin-chain

$$(3.2) \quad [A, B, C; A', B', C']$$

of type (1, 2, 3) where a, b, c are the radii of A, B, C respectively.

In view of (3.1) we may obtain a triplet (x, y, b) satisfying (2.9). Thus, if $b/c \geq 1$ (note that $b/c \geq b/a$), any point (x, y) on the "upper right-half" of the ellipse E_b (in the present discussion b is fixed) will suffice. If $b/c < 1$ then it might be necessary to solve the first two equations of (2.12), which leads to the solution of a quartic, before a

point (x, y) on the upper right-half of E , and in R may be selected. The triplet (x, y, b) satisfying (2.9) leads to a triplet (m, l, b) satisfying (2.6). Now, let numbers n and u be defined by (2.3) and (2.4), respectively. We note that $u > n > 0$; this will imply that the chain (3.2) which we construct below will be of type (1, 2, 3). (2.5) holds since the first expression of (2.6) implies

$$(3.3) \quad \{2c(b+m)\}^{1/2} + \{2a(b+l)\}^{1/2} \geq (2ac)^{1/2};$$

but the left side of (3.3) is equal to u (by (2.3) and (2.4)).

Now, let the centers of A, B, C, A', B', C' be given by (2.1). Making use of (2.6) and/or (2.3), (2.4), (2.5) it may be shown that A and A', B and B', C and C', A and B, B and C, A' and B', B' and C' are respectively tangent, and that no two of these spheres have interior points in common. (E.g., since $n^2 = 2a(b+l)$, we have $m^2 + l^2 - 2al + a^2 + n^2 = a^2 + 2ab + b^2$, which implies that A and B are tangent. Or, since $m \geq 0$, we have $2c(b+m) \geq 2c(b-m)$; but $(u-n)^2 \geq 2c(b+m) \geq 2c(b-m)$; thus $l^2 + c^2 + 2cm + m^2 + (u-n)^2 \geq b^2 + 2bc + c^2$ which implies that C and B' have no interior points in common; etc.)

We note finally that conditions (i), (ii), (iii) in the definition of a twin-chain (cf. §1) are obviously satisfied. Thus $[A, B, C; A', B', C']$ is a normal twin-chain of type (1, 2, 3).

COROLLARY 3. *Any twin-chain $[S_1, S_2, S_3; S'_1, S'_2, S'_3]$ for which (1.4) holds, has a normal equivalent of type (1, 2, 3).*

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