CONTINUOUS FUNCTIONS DEFINED ON PRODUCT-SPACES

WOLFGANG M. SCHMIDT

1. **The results.** The most concrete result of this paper is

THEOREM 1. *Let $f(x, y)$ be a continuous double-periodic function satisfying $f(x+1, y)=f(x, y+1)=f(x, y)$. Let α, β be arbitrary. Then there exist x, y, $\bar{y}$ having*

$$f(x, y) = f(x, y + \beta) = f(x + \alpha, \bar{y}) = f(x + \alpha, \bar{y} + \beta).$$

Thus f maps the vertices of a certain parallelogram into a single number. In [1] I proved the theorem in special cases and showed that it has an application to continuous functions on the 3-sphere. In [1] I also showed that the theorem would no longer be true if one would ask for $y=\bar{y}$.

More generally, we say a class σ_k of k-tuples of points in a compact topological space S has property p, if every real-valued map f of S maps all the points of a k-tuple $\Sigma_f \in \sigma_k$ into a single point. Here and throughout the paper, compact means sequentially compact. Thus if S is the n-sphere S^n and σ_{n+1} the class of orthogonal $(n+1)$-tuples on S, then the Kakutani-Yamabe-Yujobo-Theorem states that σ_{n+1} has property p.

We call the topological product of a line with S *cylinder* over S and denote it by $C(S)$. Points of $C(S)$ will be written (x, X), where x is a real number, $X \in S$. A continuous curve in $C(S)$, $x(t)$, $X(t)$, $-\infty < t < \infty$, will be called a *rain* over S if $x(t)$ tends to $\pm\infty$ when t tends to $\pm\infty$. A *roof* over S is a compact set in $C(S)$ which has a nonempty intersection with every rain over S. A class σ_k of k-tuples in S has property P if to any roof R over S there exists a k-tuple $\Sigma_R \in \sigma_k$ and an x such that

$$(x, X) \in R \quad \text{for every} \quad X \in \Sigma_R.$$

Since every real-valued map f of a compact space S is associated with the roof $(f(X), X)$, property P implies p.

Now let $X_1, \cdots, X_{n+1}$ be an $(n+1)$-tuple of points on the n-sphere S^n whose spherical distances satisfy

$$d(X_i, X_j) = d(X_1, X_j) \qquad (1 \leqq i < j \leqq n + 1).$$

Let τ_{n+1} be the class of $(n+1)$-tuples obtained by applying a rotation

Received by the editors December 13, 1960.

to our particular $X_1, \cdots, X_{n+1}$. Then the methods of Yamabe-Yujobo [2] show that τ_{n+1} has property P.

We say a sequence $\Sigma_1, \Sigma_2, \cdots$ of k-tuples is convergent to a k-tuple Σ, if the elements $X_{i1}, X_{i2}, \cdots, X_{ik}$ of Σ_i and $X_1, \cdots, X_k$ of Σ can be arranged in such a way that $\lim X_{ij} = X_j$ $(j=1, \cdots, k)$. We call a class σ_k closed if the limit of any convergent sequence of k-tuples of σ_k is again in σ_k.

If σ_k is a class of k-tuples in S and τ_l a class of l-tuples in T, then we define $\sigma \times \tau$ to be the class of the following $k \cdot l$-tuples in the topological product $S \times T$. The $k \cdot l$-tuples of $\sigma \times \tau$ consist of all pairs of the type (X, Y), where X runs through a k-tuple Σ of σ_k and, for given X, Y runs through an l-tuple T_X of τ_l. For example, if $S = T$ is the space of real numbers modulo 1 and $\sigma_2(\alpha)$ the class of pairs x, x' having $x - x' = \alpha$, then $\sigma_4(\alpha, \beta) = \sigma_2(\alpha) \times \sigma_2(\beta)$ consists of quadruples (x, y), $(x, y+\beta)$, $(x+\alpha, \bar{y})$, $(x+\alpha, \bar{y}+\beta)$.

THEOREM 2. *Assume σ_k has property P in S, τ_l has property P in T and τ_l is closed. Then $\sigma \times \tau$ has property P in $S \times T$.*

It appears to be difficult to generalize our results to maps f into R^n and to prove the following generalization of the Borsuk-Ulam Theorem: *Let $X \to -X$ be the antipodal map in S^n and let f be a map of $S^n \times S^n$ into R^n. Then there exist X, Y, $\overline{Y}$ in S^n having $f(X, Y) = f(X, -Y) = f(-X, \overline{Y}) = f(-X, -\overline{Y})$.*

2. The proofs.

LEMMA 1. *Assume R is a roof over $S \times T$ and let $x(t)$, $X(t)$ be a rain N over S. Then the set $G(N)$ of points (t, Y) of $C(T)$ where*

$$(x(t), X(t), Y) \in R$$

forms a roof over T.

PROOF. If (t_n, Y_n) is a sequence in $G(N)$, then $(x(t_n), X(t_n), Y_n) \in R$ has a subsequence convergent to some $(x, X, Y) \in R$. For this subsequence $x(t_n)$, and therefore t_n, is bounded, and t_n will have a limit-point t_0 where $x = x(t_0)$, $X = X(t_0)$. Thus (t_0, Y) will be a limit-point of (t_n, Y_n) in $G(N)$, and $G(N)$ is compact.

Thus if $G(N)$ were not a roof, there would exist a rain $t(s)$, $Y(s)$ over T, having no point in G. Then $x(t(s))$, $X(t(s))$, $Y(s)$ would be a rain over $S \times T$ with no point in R.

LEMMA 2. *Let R be a roof over $S \times T$ and assume τ_l of T is closed and has property P. Then the set H of points (x, X) in $C(S)$ such that for suitable $\mathrm{T}(x, X) \in \tau$*

$$(x, X, Y) \in R \quad \textit{for every} \quad Y \in \mathbf{T}$$

is a roof over S.

PROOF. By $R^{(l)}$ denote the set of points $(x, X, Y_1, \cdots, Y_l)$ of $C(S \times T \times \cdots \times T)$ such that $(x, X, Y_j) \in R$ $(j=1, \cdots, l)$ and $Y_1, \cdots, Y_l$ is an l-tuple of τ_l. It follows from the compactness of R and the closedness of τ_l that $R^{(l)}$ is compact. (x, X) is in H if and only if there exist $Y_1, \cdots, Y_l$ with $(x, X, Y_1, \cdots, Y_l) \in R^{(l)}$. Therefore H is compact.

Now let N be a rain over S. Then $G(N)$ is a roof over T and there exists some t and some $\mathbf{T} \in \tau$ such that $(t, Y) \in G$ for every $Y \in \mathbf{T}$. Then $(x(t), X(t), Y) \in R$ for every $Y \in \mathbf{T}$ and N has a common point with H.

PROOF OF THEOREM 2. Assume the hypotheses of the theorem to be satisfied. Construct H as in Lemma 2. By the property of σ, there exists a $\Sigma \in \sigma$ and some x such that

$$(x, X) \in H \quad \text{for every} \quad X \in \Sigma.$$

Then $(x, X, Y) \in R$ for $X \in \Sigma$, $Y \in \mathbf{T}_X$ and Theorem 2 is proved.

PROOF OF THEOREM 1. If S is the space of real numbers modulo 1 and $\sigma_2(\alpha)$ is defined as before, then $\sigma_2(\alpha)$ has property P. This is the one-dimensional case of the generalized Yamabe-Yujobo Theorem. Furthermore, σ_2 is closed. Theorem 1 is a consequence of these facts and Theorem 2.

REFERENCES

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2. H. Yamabe and Z. Yujobo, *On the continuous functions defined on a sphere*, Osaka Math. J. vol. 2 (1950) pp. 19–22.

UNIVERSITY OF COLORADO