ON F. SUPNICK'S SIX-CONIC THEOREM

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In Proc. Amer. Math. Soc. vol. 11 (1960) p. 498, F. Supnick proved that if $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, $P_6$ are any six real coplanar points, not all on the same conic and no three collinear, then it is possible for each to be inside the conic through the other five, but impossible for each point to be outside. The complete theorem is that two, three, or six of the points lie inside the conic through the other five.

We call $P_1$ an “in-point” when it lies inside “conic I” (i.e., the conic $P_2P_3P_4P_5P_6$). We call it an “out-point” when it lies outside so that the tangents from $P_1$ to conic I are real; and so for the points $P_2$, etc. Since the theorem is unaltered by any real projection, there is no loss of generality in supposing that conic I is an ellipse with the points $P_2$, $P_3$, $P_4$, $P_5$, $P_6$ in order round its perimeter. Further projections can convert the ellipse into a circle with $P_2P_4P_5P_6$ a rectangle; which we may assume to be the case in what follows.

Suppose now that $P_2$, $P_3$, $P_4$, $P_5$, $P_6$ are kept fixed and that $P_1$ traces out a continuous path in the plane. When $P_1$ is very close to the line $P_2P_3$ (but not in the immediate neighbourhood of $P_2$ or $P_3$), conic VI approximates to the line-pair $P_2P_3$, $P_4P_5$ and it will be immediately evident from the diagram whether $P_1$ is an in-point or an out-point. Also it will be clear that as $P_1$ crosses the line $P_2P_3$ each of $P_4$, $P_5$, $P_6$ changes from in-point to out-point or vice versa, while $P_1$, $P_2$ and $P_3$ do not change in this manner. Similar results hold good when $P_1$ crosses $P_2P_4$, $P_2P_6$, etc. Again it will be seen that when $P_1$ crosses the perimeter of conic I each of the six points changes from in-point to out-point or vice versa. Using these facts we can readily determine whether each of the six points is an in-point or an out-point when $P_1$ lies in any one of those portions of the plane into which it is divided by the conic I and the ten lines which join two of the five points $P_2$, $P_3$, $P_4$, $P_6$, $P_6$; and thus verify the theorem.

For instance, each of the six points is an in-point if and only if $P_1$ lies in the pentagon bounded by $P_2P_4$, $P_3P_6$, $P_4P_6$, $P_5P_2$, $P_6P_3$.  

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