In his discussion of the problem of imbedding a finite lattice into a finite partition lattice, Birkhoff [1] speculates that there are no non-trivial identities satisfied in every finite partition lattice. We give a simple proof of this conjecture based upon Whitman's Theorem [4]. Since a partition lattice $P$ is a complete, meet-continuous lattice in which every element is a join of points [3], $P$ is isomorphic to the lattice of ideals of its sublattice $P'$ of finite-dimensional elements. We prove Birkhoff's conjecture by focusing attention on $P'$ rather than on $P$ directly.

We denote the join and meet operations by $+$ and $\cdot$. A lattice polynomial form will be denoted by $f(x_1, \ldots, x_1, \ldots, x_n, \ldots, x_n)$, where we repeat a variable $m$ times if it appears $m$ times in the form.

**Lemma.** If an identity is valid in a lattice $L$, then it is valid in the lattice $L'$ of all ideals of $L$.

**Proof.** Let the identity be

$$f(x_1, \ldots, x_1, \ldots, x_n, \ldots, x_n) = g(x_1, \ldots, x_1, \ldots, x_n, \ldots, x_n).$$

If $t \in f(X_1, \ldots, X_1, \ldots, X_n, \ldots, X_n)$, then

$$t \leq f(x_{11}, \ldots, x_{1n}, \ldots, x_{n1}, \ldots, x_{nn}).$$

where $x_{pq} \in X_p$, the $X_p$ being ideals. Obviously

$$t \leq f(x'_1, \ldots, x'_1, \ldots, x'_n, \ldots, x'_n) = g(x'_1, \ldots, x'_1, \ldots, x'_n, \ldots, x'_n)$$

where $x'_i = \sum_p x_{ip}$.

Thus $t \in g(X_1, \ldots, X_1, \ldots, X_n, \ldots, X_n)$ since the $X_p$ are ideals. It therefore follows that $f(X_1, \ldots, X_1, \ldots, X_n, \ldots, X_n) \subseteq g(X_1, \ldots, X_1, \ldots, X_n, \ldots, X_n)$, and by symmetry the identity $f = g$ is valid in $L'$.

**Remark.** This lemma is an exercise (unanswered) in [2, p. 80].

**Theorem.** Any identity valid in every finite partition lattice is trivial, i.e. is valid in every lattice.

**Proof.** Let us suppose that the identity $f = g$ is valid in every finite

Received by the editors November 7, 1960.
partition lattice. Let \( P \) be any partition lattice, and let \( P' \) be its sublattice of finite-dimensional elements. The join of any \( n \) elements in \( P' \) lies in an interval sublattice \([0, a]\) which is isomorphic to a finite partition lattice. (Take the union of all the sets of the \( n \) partitions which are nonsingletons.) Thus if the identity \( f = g \) is valid in every finite partition lattice, it is valid in \( P' \). In view of the lemma and the fact that \( P \) is isomorphic to the lattice of ideals in \( P' \), the identity \( f = g \) is valid in \( P \). By Whitman’s Theorem every lattice is a sublattice of some partition lattice, and therefore the identity \( f = g \) is valid in every lattice.

**Corollary.** There are no nontrivial identities valid in any infinite partition lattice.

**Corollary.** For every nontrivial identity \( f = g \), there exists a finite lattice in which it is invalid.

The last corollary suggests the following question: If an identity is valid in every finite sublattice of a lattice \( L \), is it valid in \( L \)? This is certainly the case for the modular and distributive identities.

**Bibliography**


