IDENTITIES IN FINITE PARTITION LATTICES

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In his discussion of the problem of imbedding a finite lattice into a finite partition lattice, Birkhoff [1] speculates that there are no non-trivial identities satisfied in every finite partition lattice. We give a simple proof of this conjecture based upon Whitman's Theorem [4]. Since a partition lattice $P$ is a complete, meet-continuous lattice in which every element is a join of points [3], $P$ is isomorphic to the lattice of ideals of its sublattice $P'$ of finite-dimensional elements. We prove Birkhoff's conjecture by focusing attention on $P'$ rather than on $P$ directly.

We denote the join and meet operations by $+$ and $\cdot$. A lattice polynomial form will be denoted by $f(x_1, \ldots, x_1, \ldots, x_n, \ldots, x_n)$, where we repeat a variable $m$ times if it appears $m$ times in the form.

**Lemma.** If an identity is valid in a lattice $L$, then it is valid in the lattice $L'$ of all ideals of $L$.

**Proof.** Let the identity be

$$f(x_1, \ldots, x_1, \ldots, x_n, \ldots, x_n) = g(x_1, \ldots, x_1, \ldots, x_n, \ldots, x_n).$$

If $t \in f(X_1, \ldots, X_1, \ldots, X_n, \ldots, X_n)$, then

$$t \leq f(x_{1_1}, \ldots, x_{1_i}, \ldots, x_{1_n}, \ldots, x_{1_j})$$

where $x_{1p} \in X_p$, the $X_p$ being ideals. Obviously

$$t \leq f(x'_1, \ldots, x'_1, \ldots, x'_n, \ldots, x'_n) = g(x'_1, \ldots, x'_1, \ldots, x'_n, \ldots, x'_n)$$

where $x'_1 = \sum_p x_{1p}$.

Thus $t \in g(X_1, \ldots, X_1, \ldots, X_n, \ldots, X_n)$ since the $X_p$ are ideals. It therefore follows that $f(X_1, \ldots, X_1, \ldots, X_n, \ldots, X_n) \subseteq g(X_1, \ldots, X_1, \ldots, X_n, \ldots, X_n)$, and by symmetry the identity $f = g$ is valid in $L'$.

**Remark.** This lemma is an exercise (unanswered) in [2, p. 80].

**Theorem.** Any identity valid in every finite partition lattice is trivial, i.e. is valid in every lattice.

**Proof.** Let us suppose that the identity $f = g$ is valid in every finite
partition lattice. Let $P$ be any partition lattice, and let $P'$ be its sub-
lattice of finite-dimensional elements. The join of any $n$ elements in
$P'$ lies in an interval sublattice $[0, a]$ which is isomorphic to a finite
partition lattice. (Take the union of all the sets of the $n$ partitions
which are nonsingletons.) Thus if the identity $f = g$ is valid in every
finite partition lattice, it is valid in $P'$. In view of the lemma and the
fact that $P$ is isomorphic to the lattice of ideals in $P'$, the identity
$f = g$ is valid in $P$. By Whitman's Theorem every lattice is a sublattice
of some partition lattice, and therefore the identity $f = g$ is valid in
every lattice.

**Corollary.** There are no nontrivial identities valid in any infinite
partition lattice.

**Corollary.** For every nontrivial identity $f = g$, there exists a finite
lattice in which it is invalid.

The last corollary suggests the following question: If an identity
is valid in every finite sublattice of a lattice $L$, is it valid in $L$? This is
certainly the case for the modular and distributive identities.

**Bibliography**

1. G. Birkhoff, *Some problems of lattice theory*, International Congress of Mathe-
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