ON GALOIS SUBRINGS OF A FULL RING OF LINEAR TRANSFORMATIONS[1]

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Although several authors, including Dieudonné [**1**], Hochschild [**3**], Nakayama [**5**], Rosenberg and Zelinsky [**9**], and Jacobson [**4**], have worked on the Galois theory of simple rings with chain conditions, much of their work is concerned with 1-1 correspondences between groups of automorphisms of a simple ring and Galois subrings. Very little, however, has been done towards classifying those subrings of a simple ring which are Galois. Jacobson [**4**] gives a condition due to Nakayama for a weakly Galois completely reducible subring to be Galois. In this paper we give a sufficient condition for a non-necessarily completely reducible weakly Galois subring to be Galois. Our result overlaps Nakayama's but does not properly contain his.

Throughout this paper we shall use the following notations: Δ, a division ring finite dimensional over its center Γ, $\Gamma \neq GF(2)$; M, a finitely generated Δ-module; $L = \mathrm{Hom}_\Delta(M, M)$; R, a subring of L containing Γ. We assume that M as a module over any ring considered is a left-module.

Since M is a finite dimensional vector space over Δ, it is also a finite dimensional vector space over Γ. Thus $\mathrm{Hom}_\Gamma(M, M)$ can be regarded in the usual way as a finitely generated algebra over Γ. And since the rings R, Δ, Γ, L, and ΔR are contained in $\mathrm{Hom}_\Gamma(M, M)$, they are all finite dimensional over Γ and thus all these rings satisfy all the chain conditions and M is a finitely generated module over each of them.

LEMMA 1.[2] *If Δ is a division ring finite dimensional over its center Γ, if R is any Γ-algebra and if M is an $(R\otimes\Delta)$-module which is injective as an R-module, then M is also injective as an $(R\otimes\Delta)$-module.*

PROOF. This follows immediately from [**2**, Proposition 2, p. 74], which asserts l. inj. $\dim_{R\otimes\Delta} M \leqq \dim \Delta +$l. inj. $\dim_R M$, because $\dim \Delta = 0$ since Δ is separable over Γ and l. inj. dim. is zero wherever the module is injective and conversely.

Presented to the Society, November 28, 1959; received by the editors September 12, 1960.

[1] Part of the research on this paper was done while the author was a National Science Foundation Pre-doctoral Fellow and the writing of this paper was supported in part by the Office of Naval Research.

[2] The author wishes to thank the referee for his suggestions in general and in particular for calling the proof of this lemma to the author's attention.

LEMMA 2. *If R is a quasi-Frobenius subring of* $\mathrm{Hom}_\Delta(M, M)$, $R \supseteq \Gamma$, *then* $\Delta R \cong$ *(isomorphic)* $\Delta \otimes_\Gamma R$ *and* ΔR *is quasi-Frobenius.*

PROOF. By [4, Corollary 5.6, 1], $\Delta R \cong \Delta \otimes_\Gamma R$ if R and Δ commute, if Δ is central simple and if no nonzero element of Δ annihilates R. These conditions are satisfied here. By [7, Theorem 14], the tensor product of quasi-Frobenius algebras of finite dimension is quasi-Frobenius.

LEMMA 3. *If R is quasi-Frobenius and if* $R' = \mathrm{Hom}_R(M, M)$ *is generated over Δ by* 1-1 *Δ-semi-linear transformations of M, then R is Galois in L.*

PROOF. Let S be the set of semi-linear transformations in R' with inverses. Clearly, $R' = \Delta S$. Let G be the group of inner automorphisms of $\mathrm{Hom}_\Gamma(M, M)$ produced by elements of S. The group G induces a group of automorphisms on L whose fixed ring is $F = \{x \mid xs = sx$ for all s in $S\}$ = centralizer of $\Delta S = \mathrm{Hom}_{R'}(M, M)$. Since R is quasi-Frobenius, $\mathrm{Hom}_{R'}(M, M) = R$ by the double-centralizer theorem [8], so $F = R$ and R is the fixed ring under the restriction of G to L.

THEOREM. *If R is quasi-Frobenius, if M is an injective R-module, and if R is weakly Galois in L, then R is Galois in L.*

PROOF. By Lemma 3, it suffices to prove that $\mathrm{Hom}_R(M, M)$ is generated over Δ by 1-1 Δ-semi-linear transformations of M.

Decompose M into a direct sum $\sum^{\oplus} M_i$ of indecomposable ΔR-modules. By the above lemmas, the M_i are all injective.

CASE 1. s is a semi-linear transformation which is zero on $\sum^{\oplus}_{i>1} M_i$ and has kernel zero on M_1. Since s is semi-linear, it corresponds to some automorphism of Δ over Γ, which must be an inner automorphism because Δ is finite-dimensional and central simple over Γ. Thus there is a $\mu \neq 0$ in Δ such that for all δ in Δ, $s\delta = (\mu^{-1}\delta\mu)s$. Thus, μs commutes with Δ as well as with R, μs has kernel zero on M_1 and so μs gives a ΔR-isomorphism of M_1 onto $\mu s(M_1) = s(M_1)$. This proves that $s(M_1)$ is indecomposable and injective over ΔR. We decompose M a second time: $M = s(M_1) \oplus \sum^{\oplus}_{i>1} M_i'$. By Krull-Schmidt, we can find isomorphisms ϕ_i of M_i onto M_i' $(i = 2, \cdots, n)$. Since $\Gamma \neq \mathrm{GF}(2)$, choose $\gamma \neq 0, 1$ in Γ. Define s_1 to be γs on M_1, $\mu^{-1}\phi_i$ on M_i $(i > 1)$ and define s_2 to be $(1-\gamma)s$ on M_1, $-\mu^{-1}\phi_i$ on M_i, so that $s = s_1 + s_2$ and each of s_1, s_2 is a semi-linear transformation with inverse.

CASE 2. $s = 0$ on $\sum^{\oplus}_{i>1} M_i$ and $(\ker s) \cap M_1 \neq 0$. Then $(\ker s) \cap M_1$ must contain the unique minimal submodule of M_1. Define $s_1 = s + \mu^{-1}$ on M_1 (μ as in Case 1) and zero on $\sum^{\oplus}_{i>1} M_i$, define $s_2 = -\mu^{-1}$ on M_1 and zero on $\sum^{\oplus}_{i>1} M_i$. Then $s = s_1 + s_2$ and s_1 and s_2 each belong to Case 1.

CASE 3. $s=s_1+\cdots+s_n$, where s_j coincides with s on M_j and is zero on $\sum_{i\neq j}^{\oplus} M_i$. By Case 2 (with 1 changed to j), s_j is a sum of semi-linear transformations in R' with inverses. Thus so is s.

Since any s in $\mathrm{Hom}_R(M, M)$ must belong to one of the above cases, the theorem is proved.

COROLLARY. *If R and R' are quasi-Frobenius, $R'\supseteq R\supseteq\Gamma$, $R'\subseteq L$, if M is injective as both an R- and R'-module, and if R is weakly Galois in L, then R and R' are Galois in L.*

PROOF. R is Galois by the Theorem. That R' is weakly Galois if R is weakly Galois is well known (see, for example, [**4**, p. 149, proof of Fundamental Theorem]), so R' also satisfies the hypotheses of the theorem.

REMARK 1. If M is a direct sum of isomorphic indecomposable R-modules and if R is quasi-Frobenius, then M is an injective R-module.

PROOF. M contains as a direct summand the reduced regular representation of R. Thus, all the direct summands of M are isomorphic to Re, e a primitive idempotent in R. It is well known that Re is injective.

REMARK 2. If R is a principal ideal subring of L, then R is quasi-Frobenius. This follows immediately from [**6**, Theorem 2, p. 285 and Lemma 2, p. 286]. A special case is the ring $\Gamma[T]$ generated over Γ by a single Δ-linear transformation T of M.

This paper formed part of a doctoral dissertation written under Professor Charles W. Curtis at the University of Wisconsin. The author wishes to express his deep appreciation to Professor Curtis for his many valuable suggestions.

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