THE EXISTENCE OF COMPLETE RIEMANNIAN METRICS

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The purpose of the present note is to prove the following results. Let M be a connected differentiable manifold which satisfies the second axiom of countability. Then (i) M admits a complete Riemannian metric; (ii) If every Riemannian metric on M is complete, M must be compact.

In fact, somewhat stronger results will be given as Theorems 1 and 2 below.

Let M be a connected differentiable manifold. It is known that if M satisfies the second axiom of countability, then M admits a Riemannian metric. Conversely, it can be shown that the existence of a Riemannian metric on M implies that M satisfies the countability axiom. For any Riemannian metric g on M, we can define a natural metric d on M by setting the distance d(x, y) between two points x and y to be the infinimum of the lengths of all piecewise differentiable curves joining x and y. The Riemannian metric g is complete if the metric space M with d is complete. It is known that this is the case if and only if every bounded subset of M (with respect to d) is relatively compact.

We shall say that a Riemannian metric g is bounded if M is bounded with respect to the metric d. We shall prove

Theorem 1. For any Riemannian metric g on M, there exists a complete Riemannian metric which is conformal to g.

Theorem 2. For any Riemannian metric g on M, there exists a bounded Riemannian metric which is conformal to g.

The result (ii) mentioned in the beginning is a consequence of Theorem 2, because if a bounded Riemannian metric, which exists on M, is complete, then M itself is compact.

Proof of Theorem 1. At each point x of M, we define r(x) to be the supremum of positive numbers r such that the neighborhood S(x, r) = {y; d(x, y) < r} is relatively compact. If r(x) = ∞ at some point x, M is compact and hence g is complete. Assume therefore that r(x) < ∞ for every x. It is easy to verify that |r(x) − r(y)| ≤ d(x, y) for all x and y in M, which shows that r(x) is a continuous function on

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$M$. Since $M$ satisfies the second axiom of countability, we can choose a differentiable function $\omega(x)$ such that $\omega(x) > 1/r(x)$ at every point $x$. We define a conformal Riemannian metric $g'$ by $g'_x = (\omega(x))^2 g_x$ at every point $x$.

In order to prove that $g'$ is complete, we shall show that $S'(x, 1/3) = \{ y; d'(x, y) < 1/3 \}$ is contained in $S(x, r(x)/2)$ (and hence relatively compact) for every $x$, where $d'$ is the distance defined by $g'$. For this purpose, assume $d(x, y) \geq r(x)/2$. For any piecewise differentiable curve $x(t), a \leq t \leq b$, joining $x$ and $y$, its length $L = \int_a^b \| dx/dt \| dt \| dx/dt \|$ denotes the length of the tangent vector $dx/dt$ with respect to $g$ is not smaller than $d(x, y)$ and hence $L \geq r(x)/2$. We evaluate the length $L'$ of the same curve with respect to $g'$. By a mean value theorem, we have

$$L' = \int_a^b \omega(x) \| dx/dt \| dt \omega(x(c)) L \geq L/r(x(c)),$$

where $c$ is a number between $a$ and $b$. Since $|r(x(c)) - r(x)| < d(x, x(c)) \leq L$, we have $r(x(c)) < r(x) + L$ so that $L' > L/(r(x) + L)$. Since $L \geq r(x)/2$, we have $L' > 1/3$. Therefore $d'(x, y) \geq 1/3$. This proves that $S'(x, 1/3)$ is contained in $S(x, r(x)/2)$.

Proof of Theorem 2. By virtue of Theorem 1, we may assume that the given Riemannian metric $g$ is complete. Let $o$ be an arbitrarily fixed point of $M$. The function $d(x, o)$ is continuous. Let $\omega(x)$ be a differentiable function such that $\omega(x) > d(x, o)$ on $M$. We shall prove that the Riemannian metric $g' = e^{-2\omega(o)} g$ is bounded. Let $x$ be an arbitrary point of $M$. Since $g$ is complete, there exists a minimizing geodesic $C$ from $o$ to $x$, that is, a geodesic $C$ whose length $L$ is equal to $d(x, o)$. Let $x(s)$ be a parametric representation of $C$ in terms of the arc length measured from $o$. Since any subarc of $C$ is a minimizing geodesic between its end points, we have $d(x(s), o) = s$ for every $s$. The length of the tangent vector $dx/ds$ with respect to $g'$ is equal to

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2 This fact, mentioned in the introduction, can be proved, for example, as follows. When $M$ is not compact, we define for every natural number $n$ a neighborhood $U_n(x) = \{ y; d(x, y) < r(x)/n \}$ for each point $x$. It is easy to verify that $\{ U_n \}$ defines a uniform structure on the space $M$ and that $M$ is uniformly locally compact (i.e., there is some $n$, indeed $n = 2$ will do in this case, such that $U_n(x)$ is relatively compact for every $x$). Since $M$ is connected, it follows that $M$ is the sum of countably many compact subsets. Now for a differentiable manifold, this means that it satisfies the second axiom of countability.
$e^{-u(x(s))}$. The length $L'$ of $C$ with respect to $g'$ is thus $\int_0^L e^{-u(x(s))} ds$. Since $\omega(x(s)) - d(x(s), o) = s$, we have

$$L' < \int_0^L e^{-s} ds < \int_0^\infty e^{-s} ds = 1,$$

which implies that $d'(x, o) < 1$ for every $x$.

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