THE EXISTENCE OF COMPLETE RIEMANNIAN METRICS

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The purpose of the present note is to prove the following results. Let \(M\) be a connected differentiable manifold which satisfies the second axiom of countability. Then (i) \(M\) admits a complete Riemannian metric; (ii) If every Riemannian metric on \(M\) is complete, \(M\) must be compact.

In fact, somewhat stronger results will be given as Theorems 1 and 2 below.

Let \(M\) be a connected differentiable manifold. It is known that if \(M\) satisfies the second axiom of countability, then \(M\) admits a Riemannian metric. Conversely, it can be shown that the existence of a Riemannian metric on \(M\) implies that \(M\) satisfies the countability axiom. For any Riemannian metric \(g\) on \(M\), we can define a natural metric \(d\) on \(M\) by setting the distance \(d(x, y)\) between two points \(x\) and \(y\) to be the infimum of the lengths of all piecewise differentiable curves joining \(x\) and \(y\). The Riemannian metric \(g\) is complete if the metric space \(M\) with \(d\) is complete. It is known that this is the case if and only if every bounded subset of \(M\) (with respect to \(d\)) is relatively compact.

We shall say that a Riemannian metric \(g\) is bounded if \(M\) is bounded with respect to the metric \(d\). We shall prove

**Theorem 1.** For any Riemannian metric \(g\) on \(M\), there exists a complete Riemannian metric which is conformal to \(g\).

**Theorem 2.** For any Riemannian metric \(g\) on \(M\), there exists a bounded Riemannian metric which is conformal to \(g\).

The result (ii) mentioned in the beginning is a consequence of Theorem 2, because if a bounded Riemannian metric, which exists on \(M\), is complete, then \(M\) itself is compact.

**Proof of Theorem 1.** At each point \(x\) of \(M\), we define \(r(x)\) to be the supremum of positive numbers \(r\) such that the neighborhood \(S(x, r) = \{y; d(x, y) < r\}\) is relatively compact. If \(r(x) = \infty\) at some point \(x\), \(M\) is compact and hence \(g\) is complete. Assume therefore that \(r(x) < \infty\) for every \(x\). It is easy to verify that \(|r(x) - r(y)| \leq d(x, y)\) for all \(x\) and \(y\) in \(M\), which shows that \(r(x)\) is a continuous function on

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$M$. Since $M$ satisfies the second axiom of countability, we can choose a differentiable function $\omega(x)$ such that $\omega(x) > 1/r(x)$ at every point $x$. We define a conformal Riemannian metric $g'$ by $g' = (\omega(x))^2 g_\ast$ at every point $x$.

In order to prove that $g'$ is complete, we shall show that $S'(x, 1/3) = \{ y; d'(x, y) < 1/3 \}$ is contained in $S(x, r(x)/2)$ (and hence relatively compact) for every $x$, where $d'$ is the distance defined by $g'$. For this purpose, assume $d(x, y) \geq r(x)/2$. For any piecewise differentiable curve $x(t), a \leq t \leq b$, joining $x$ and $y$, its length $L = \int_a^b ||dx/dt|| dt$ denotes the length of the tangent vector $dx/dt$ with respect to $g$ is not smaller than $d(x, y)$ and hence $L \geq r(x)/2$. We evaluate the length $L'$ of the same curve with respect to $g'$. By a mean value theorem, we have

$$L' = \int_a^b \omega(x)||dx/dt|| dt \omega(x(c))L$$

$$> L/r(x(c)),$$

where $c$ is a number between $a$ and $b$. Since $|r(x(c)) - r(x)| < d(x, x(c)) \leq L$, we have $r(x(c)) < r(x) + L$ so that $L' > L/(r(x) + L)$. Since $L \geq r(x)/2$, we have $L' > 1/3$. Therefore $d'(x, y) \geq 1/3$. This proves that $S'(x, 1/3)$ is contained in $S(x, r(x)/2)$.

Proof of Theorem 2. By virtue of Theorem 1, we may assume that the given Riemannian metric $g$ is complete. Let $o$ be an arbitrarily fixed point of $M$. The function $d(x, o)$ is continuous. Let $\omega(x)$ be a differentiable function such that $\omega(x) > d(x, o) on M$. We shall prove that the Riemannian metric $g' = e^{-2\omega(x)} g$ is bounded. Let $x$ be an arbitrary point of $M$. Since $g$ is complete, there exists a minimizing geodesic $C$ from $o$ to $x$, that is, a geodesic $C$ whose length $L$ is equal to $d(x, o)$. Let $x(s)$ be a parametric representation of $C$ in terms of the arc length measured from $o$. Since any subarc of $C$ is a minimizing geodesic between its end points, we have $d(x(s), o) = s$ for every $s$. The length of the tangent vector $dx/ds$ with respect to $g'$ is equal to

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\footnote{This fact, mentioned in the introduction, can be proved, for example, as follows. When $M$ is not compact, we define for every natural number $n$ a neighborhood $U_n(x) = \{ y; d(x, y) < r(x)/n \}$ for each point $x$. It is easy to verify that $\{ U_n \}$ defines a uniform structure on the space $M$ and that $M$ is uniformly locally compact (i.e., there is some $n$, indeed $n = 2$ will do in this case, such that $U_n(x)$ is relatively compact for every $x$). Since $M$ is connected, it follows that $M$ is the sum of countably many compact subsets. Now for a differentiable manifold, this means that it satisfies the second axiom of countability.}
e^{-u(x(s))}. The length $L'$ of $C$ with respect to $g'$ is thus $\int_0^L e^{-u(x(s))} ds$. Since $\omega(x(s)) > d(x(s), o) = s$, we have

$$L' < \int_0^L e^{-s} ds < \int_0^\infty e^{-s} ds = 1,$$

which implies that $d'(x, o) < 1$ for every $x$.

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