ON THE POLE AND ZERO LOCATIONS OF RATIONAL LAPLACE TRANSFORMATIONS OF NON-NEGATIVE FUNCTIONS. II

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Let $a(t)$ be a real, bounded function of the real variable t defined in the interval, $0 \leqq t < \infty$, and let its Laplace-Stieltjes transform,

$$F(s) = \int_0^\infty e^{-st} da(t), \tag{1}$$

be a rational function of the complex variable, $s = \sigma + i\omega$, of the form

$$F(s) = \frac{\prod_{i=1}^{h} (s - \eta_i) \prod_{i=1}^{g} (s - \delta_i)}{\prod_{i=1}^{m} (s - \rho_i) \prod_{i=1}^{q} (s - \xi_i)}. \tag{2}$$

In (2), the η_i and δ_i represent the real and complex zeros, respectively, and the ρ_i and ξ_i represent the real and complex poles, respectively. The complex poles and zeros will occur in complex conjugate pairs and the poles will have non positive real parts.

Various sets of sufficient conditions on the poles and zeros, which insure that $a(t)$ is nondecreasing, have been published previously [**1–5**]. The purpose of this note is to add to this body of results an extension of the theorem given in [**5**]. It includes as special cases most of the results given in [**1**; **2**; and **5**]. It does not, however, include the results of [**3**; **4**].

The following notation and numbering system will be used for the poles and zeros. The real poles are numbered according to their decreasing values; that is, $\rho_1 \geqq \rho_2 \geqq \cdots \geqq \rho_m$. The real parts of the complex poles and of all the zeros are denoted by α_i and numbered according to their decreasing values; that is, $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n$ where $n = h + g + q$. When several zeros and complex poles have the same real part, they are numbered in any order. If the multiplicity of any pole or zero is r, it is counted r times.

The following fact is an immediate consequence of the first lemma and the last four sentences of [**5**].

LEMMA 1. *If the poles and zeros of $F(s)$ are such that $\alpha_i \leqq \rho_1 < 0$ $(i = 1, 2, \cdots, n)$ and*

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$$\sum_{i=1}^{m} \frac{1}{(\sigma - \rho_i)^k} \geqq \sum_{i=1}^{n} \frac{1}{(\sigma - \alpha_i)^k} \tag{3}$$

where $\sigma \geqq 0$, $m \geqq n = h + g + q$, *and* $k = 1, 2, 3, \cdots$, *then* $a(t)$ *is non-decreasing.*

We shall also require

LEMMA 2. *Let*

$$\begin{aligned} x_1 \geqq x_2 \geqq \cdots \geqq x_{q_1} \geqq y_1 \geqq y_2 \geqq \cdots \geqq y_{p_1} \\ \geqq x_{q_1+1} \geqq \cdots \geqq x_{q_2} \geqq y_{p_1+1} \geqq \cdots \geqq y_{p_2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \geqq x_{q_{\nu-1}+1} \geqq \cdots \geqq x_{q_\nu} \geqq y_{p_{\nu-1}+1} \geqq \cdots \geqq y_{p_\nu} \geqq 0 \end{aligned}$$

where $x_{q_\nu} > 0$. *Let*

$$\sum_{i=1}^{q_\mu} x_i \geqq \sum_{i=1}^{p_\mu} y_i \qquad (\mu = 1, 2, \cdots, \nu). \tag{4}$$

Then

$$\sum_{i=1}^{q_\nu} (x_i + 1)^k + p_\nu - q_\nu \geqq \sum_{i=1}^{p_\nu} (y_i + 1)^k \tag{5}$$

where $k = 1, 2, 3, \cdots$.

PROOF. Let $u_i = x_i / x_{q_\nu} \geqq 1$ and $v_i = y_i / x_{q_\nu}$. It will first be established that

$$0 \leqq \sum_{i=1}^{q_\mu} u_i - \sum_{i=1}^{p_\mu} v_i \leqq \sum_{i=1}^{q_\mu} u_i^k - \sum_{i=1}^{p_\mu} v_i^k, \qquad (\mu = 1, 2, \cdots, \nu). \tag{6}$$

For $\mu = 1$, this follows immediately from (4). Now, assume that (6) holds for a particular $\mu \leqq \nu - 1$. From (4),

$$\sum_{i=1}^{q_{\mu+1}} \frac{u_i}{u_{q_{\mu+1}}} - \sum_{i=1}^{p_{\mu+1}} \frac{v_i}{u_{q_{\mu+1}}} \geqq 0. \tag{7}$$

Since $0 \leqq v_i / u_{q_{\mu+1}} \leqq 1$ for $p_\mu + 1 \leqq i \leqq p_{\mu+1}$ and $u_i / u_{q_{\mu+1}} \geqq 1$ for $i \leqq q_{\mu+1}$,

$$\sum_{i=q_\mu+1}^{q_{\mu+1}} \frac{u_i}{u_{q_{\mu+1}}} - \sum_{i=p_\mu+1}^{p_{\mu+1}} \frac{v_i}{u_{q_{\mu+1}}} \leqq \sum_{i=q_\mu+1}^{q_{\mu+1}} \frac{u_i^k}{u_{q_{\mu+1}}^k} - \sum_{i=p_\mu+1}^{p_{\mu+1}} \frac{v_i^k}{u_{p_{\mu+1}}^k}. \tag{8}$$

Replacing u_i by $u_i / u_{q_{\mu+1}}$ and v_i by $v_i / u_{q_{\mu+1}}$ in (6), we have by assumption that (8) holds even when the summations are taken from $i = 1$ to

$i=q_\mu$ for the terms involving u_i and from $i=1$ to $i=p_\mu$ for the terms involving v_i. Combining this result and (8) with (7), we obtain

$$0 \leqq \sum_{i=1}^{q_{\mu+1}} u_i - \sum_{i=1}^{p_{\mu+1}} v_i \leqq \frac{1}{u_{q_{\mu+1}}^{k-1}} \left[\sum_{i=1}^{q_{\mu+1}} u_i^k - \sum_{i=1}^{p_{\mu+1}} v_i^k \right]. \tag{9}$$

Since $u_{q_{\mu+1}} \geqq 1$, multiplication of the right-hand side of (9) by $u_{q_{\mu+1}}^{k-1}$ yields (6) with μ replaced by $\mu+1$. Hence, (6) holds for $\mu=\nu$ by induction. From (6), we have

$$\sum_{i=1}^{q_\nu} x_i^k \geqq \sum_{i=1}^{p_\nu} y_i^k$$

which implies (5).

The principal result of this note is given by the

THEOREM. *Let $a(t)$ be a real, bounded function of the real variable t defined in the interval, $0 \leqq t < \infty$, and let its Laplace-Stieltjes transform $F(s)$ be a rational function as given by* (2). *Let $m \geqq n = h+g+q$, let $0 > \rho_1 \geqq \alpha_i \geqq \rho_n$ $(i=1, 2, \cdots, n)$, and let*

$$\begin{aligned} 0 > \rho_1 \geqq \rho_2 \geqq \cdots \geqq \rho_{q_1} \geqq \alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_{p_1} \\ \geqq \rho_{q_1+1} \geqq \cdots \geqq \rho_{q_2} \geqq \alpha_{p_1+1} \geqq \cdots \geqq \alpha_{p_2} \geqq \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \geqq \alpha_{p_{\nu-1}+1} \geqq \cdots \geqq \alpha_{p_\nu} \geqq \rho_{q_\nu+1} \geqq \cdots \geqq \rho_{q_{\nu+1}} \end{aligned}$$

where $p_\nu = q_{\nu+1} = n$. Let

$$\sum_{i=1}^{p_\mu} \alpha_i \leqq \sum_{i=1}^{q_\mu} \rho_i + (p_\mu - q_\mu)\rho_n \qquad (\mu = 1, 2, \cdots, \nu). \tag{10}$$

Then $a(t)$ is nondecreasing.

PROOF. Because of Lemma 1, it need merely be shown that (3) holds for $\sigma \geqq 0$ and for all positive integers k.

We shall first establish by induction that

$$\sum_{i=1}^{q_\mu} \frac{\rho_i - \rho_n}{\sigma - \rho_i} \geqq \sum_{i=1}^{p_\mu} \frac{\alpha_i - \rho_n}{\sigma - \alpha_i} \tag{11}$$

for $\sigma \geqq 0$ and $\mu = 1, 2, \cdots, \nu$. For any particular $\mu \leqq \nu - 1$, (10) yields

$$\sum_{i=1}^{q_\mu} (\rho_i - \rho_n) \geqq \sum_{i=1}^{p_\mu} (\alpha_i - \rho_n). \tag{12}$$

Assume that the following expression holds for $\sigma \geqq 0$. (It obviously holds for $\mu = 1$.)

$$(13)\qquad 0 \leqq \sum_{i=1}^{q_\mu} \frac{\rho_i - \rho_n}{\sigma - \rho_{q_\mu}} - \sum_{i=1}^{p_\mu} \frac{\alpha_i - \rho_n}{\sigma - \rho_{q_\mu}} \leqq \sum_{i=1}^{q_\mu} \frac{\rho_i - \rho_n}{\sigma - \rho_i} - \sum_{i=1}^{p_\mu} \frac{\alpha_i - \rho_n}{\sigma - \alpha_i}.$$

Since $\rho_{q_{\mu+1}} \leqq \rho_{q_\mu} < 0$, (12) and (13) imply

$$(14)\qquad 0 \leqq \sum_{i=1}^{q_\mu} \frac{\rho_i - \rho_n}{\sigma - \rho_{q_{\mu+1}}} - \sum_{i=1}^{p_\mu} \frac{\alpha_i - \rho_n}{\sigma - \rho_{q_{\mu+1}}} \leqq \sum_{i=1}^{q_\mu} \frac{\rho_i - \rho_n}{\sigma - \rho_i} - \sum_{i=1}^{p_\mu} \frac{\alpha_i - \rho_n}{\sigma - \alpha_i}.$$

Furthermore,

$$(15)\qquad \begin{aligned} &\sum_{i=q_\mu+1}^{q_{\mu+1}} \frac{\rho_i - \rho_n}{\sigma - \rho_{q_{\mu+1}}} - \sum_{i=p_\mu+1}^{p_{\mu+1}} \frac{\alpha_i - \rho_n}{\sigma - \rho_{q_{\mu+1}}} \\ &\qquad \leqq \sum_{i=q_\mu+1}^{q_{\mu+1}} \frac{\rho_i - \rho_n}{\sigma - \rho_i} - \sum_{i=p_\mu+1}^{p_{\mu+1}} \frac{\alpha_i - \rho_n}{\sigma - \alpha_i} \end{aligned}$$

since each positive term on the left-hand side of (15) is replaced by a larger term on the right-hand side and each negative term on the left-hand side is replaced by a negative term of smaller magnitude on the right-hand side. Replacing μ by $\mu+1$ in (10), we may write

$$(16)\qquad \begin{aligned} 0 \leqq &\sum_{i=1}^{q_\mu} \frac{\rho_i - \rho_n}{\sigma - \rho_{q_{\mu+1}}} - \sum_{i=1}^{p_\mu} \frac{\alpha_i - \rho_n}{\sigma - \rho_{q_{\mu+1}}} \\ &+ \sum_{i=q_\mu+1}^{q_{\mu+1}} \frac{\rho_i - \rho_n}{\sigma - \rho_{q_{\mu+1}}} - \sum_{i=p_\mu+1}^{p_{\mu+1}} \frac{\alpha_i - \rho_n}{\sigma - \rho_{q_{\mu+1}}}. \end{aligned}$$

Using (14) and (15), (16) implies (13) with μ replaced by $\mu+1$. Hence, (11) is established by induction.

Now, let

$$x_i = \frac{\rho_i - \rho_n}{\sigma - \rho_i}, \qquad y_i = \frac{\alpha_i - \rho_n}{\sigma - \alpha_i},$$

and invoke Lemma 2. For any positive integer k,

$$\sum_{i=1}^{p_\nu} \left(\frac{\alpha_i - \rho_n}{\sigma - \alpha_i} + 1\right)^k \leqq \sum_{i=1}^{q_\nu} \left(\frac{\rho_i - \rho_n}{\sigma - \rho_i} + 1\right)^k + p_\nu - q_\nu.$$

Noting that $p_\nu = n$ by hypothesis, we have

$$\sum_{i=1}^{n} \frac{1}{(\sigma - \alpha_i)^k} \leqq \sum_{i=1}^{q_\nu} \frac{1}{(\sigma - \rho_i)^k} + \frac{n - q_\nu}{(\sigma - \rho_n)^k} \leqq \sum_{i=1}^{n} \frac{1}{(\sigma - \rho_i)^k}.$$

It follows that (3) holds and the theorem is proven.

A similar result may be stated when $\alpha_n < \rho_n$. In this case, $p_\nu = q_\nu = n$, $(p_\mu - q_\mu)\alpha_n$ replaces $(p_\mu - q_\mu)\rho_n$ in (10), and α_n replaces ρ_n throughout the preceding proof.

The following corollaries may be stated.

COROLLARY. *Let $F(s)$ and $G(s)$ be the Laplace-Stieltjes transforms of real, bounded functions. Let $F(s)$ satisfy the hypothesis of the theorem and let $G(s)$ be obtained from $F(s)$ by decreasing the real parts of some or all of the zeros and complex poles of $F(s)$ and/or by increasing some or all of the real poles. Then the inverse Laplace-Stieltjes transform of $G(s)$ is nondecreasing.*

COROLLARY. *The theorem remains true when the restrictions that $\rho_1 < 0$ and that $a(t)$ is bounded for all t are both deleted from the hypothesis.*

The product of two Laplace-Stieltjes transforms of nondecreasing functions is also the transform of a nondecreasing function. Hence, the following question arises. Can a rational transform that satisfies the hypothesis of the theorem always be factored into simpler functions whose inverse transforms can be shown to be nondecreasing through some previously published criteria? The following is an example of a transform that satisfies the theorem.

$$F(s) = \frac{(s + 2.5)^4(s + 3.5)^2}{(s + 1)^2(s + 3)(s + 4)^3}.$$

Considering all possible combinations of the factors of this function, it has not been found possible to show that its inverse transform is nondecreasing by using any of the results of [1–5]. Hence, the theorem of this note appears to be new.

REFERENCES

1. I. F. Macdiarmid, *Transient response conditions for monotonic characteristics*, Wireless Engineer vol. 28 (1951) pp. 330–334.

2. O. P. D. Cutteridge, *Monotonic transient response*, Proc. Inst. Elec. Engrs. Part IV vol. 101 (1954) pp. 46–54.

3. E. Lukacs and O. Szász, *Some nonnegative trigonometric polynomials connected with a problem in probability*, J. Res. Nat. Bur. Standards vol. 48 (1952) pp. 139–146.

4. ———, *Nonnegative trigonometric polynomials and certain rational characteristic functions*, J. Res. Nat. Bur. Standards vol. 52 (1954) pp. 153–160.

5. A. H. Zemanian, *On the pole and zero locations of rational Laplace transformations of nonnegative functions*, Proc. Amer. Math. Soc. vol. 10 (1959) pp. 868–872.

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