Baire spaces cannot be removed. If $Q$ is the space of rationals in $E_1$ with the relative topology, there is a separately continuous $f: Q \times Q \rightarrow E_1$ which is zero on a dense subset of $Q \times Q$ but not identically zero.

**Reference**


**Purdue University**

---

**COMPLETE SEQUENCES OF FUNCTIONS\(^1\)**

CASPER GOFFMAN

Although the result of this note is implicitly contained in the work of A. A. Talalyan [2] and could also have been a corollary to the theorem in [1], it seems to be of sufficient interest to merit explicit treatment.

It is known (see [1]) that if \( \{f_1, f_2, \ldots, f_n, \ldots\} \) is a sequence of measurable functions which is complete in the space $M$ of measurable functions (i.e., every measurable $f$ is the limit in measure of a sequence of finite linear combinations of \( \{f_1, f_2, \ldots, f_n, \ldots\} \)) then \( \{f_2, f_3, \ldots, f_n, \ldots\} \) is also complete in $M$.

Let $X$ be a separable Banach space of measurable functions on $[a, b]$ such that for every measurable $G \subset [a, b]$, with $m(G) > 0$, the set $X_G$ of restrictions of the functions in $X$ to $G$ is a Banach space and

(a) If \( \{g_n\} \) converges to $g$ in $X$ then \( \{g_n\} \) converges to $g$ in $X_G$,

(b) The set of bounded measurable functions is a dense subset of $X_G$;

(c) For every $G$, uniform convergence on $G$ implies convergence in $X_G$ and convergence in $X_G$ implies convergence in measure on $G$.

**Theorem.** If \( \{f_1, f_2, \ldots, f_n, \ldots\} \) is complete in $X$ and $\varepsilon > 0$, there is a measurable $G \subset [a, b]$, with $m(G) > (b-a) - \varepsilon$, such that \( \{f_2, f_3, \ldots, f_n, \ldots\} \) is complete in $X_G$.

**Proof.** Let \( \{g_1, g_2, \ldots, g_n, \ldots\} \) be dense in $X$. Since \( \{f_1, f_2, \ldots, f_n, \ldots\} \) is complete in $X$, it follows from (b), (c) and

---

1 Supported by National Science Foundation Grant NSF-G18920.
the fact that the bounded functions are dense in \( M \), that
\[ \{f_1, f_2, \ldots, f_m, \ldots \} \]
is complete in \( M \). It follows from [1] that for every \( n \), there is a sequence \( \{\phi_1, \phi_2, \ldots, \phi_m, \ldots \} \) of finite linear combinations of \( \{f_2, f_3, \ldots, f_n, \ldots \} \) which converges in measure to \( g_n \), and so has a subsequence \( \{\psi_1, \psi_2, \ldots, \psi_m, \ldots \} \) which converges uniformly to \( g_n \) on a measurable set \( G_n \), with \( m(G_n) > (b-a) - \epsilon/2^n \).

Let \( G = \bigcap_{n=1}^{\infty} G_n \). Since uniform convergence on \( G \) implies convergence in \( X_\sigma \) by (a), and since \( \{g_1, g_2, \ldots, g_n, \ldots \} \) is dense in \( X_\sigma \) by (c), it follows that \( \{f_2, f_3, \ldots, f_n, \ldots \} \) is complete in \( X_\sigma \).

If \( X = L_2[a, b] \), then \( X_\sigma = L_2(G) \), so that we have:

**Corollary.** If \( \{f_1, f_2, \ldots, f_n, \ldots \} \) is complete for \( L_2[a, b] \) and \( \epsilon > 0 \) there is a measurable \( G \subset [a, b] \), \( m(G) > (b-a) - \epsilon \) such that \( \{f_2, f_3, \ldots, f_n, \ldots \} \) is complete for \( L_2(G) \).

**References**
