ON QUASI-REFLEXIVE BANACH SPACES

Y. CUTTLE

1. Introduction. In the first part of this paper we study some properties of Schauder bases in quasi-reflexive Banach spaces. It is shown that if a quasi-reflexive space of order 1 has a Schauder basis, this basis must be either shrinking or boundedly complete. This result should be compared with the well-known theorem which states that if a reflexive Banach space has a Schauder basis, then this basis must be both shrinking and boundedly complete. If a quasi-reflexive space has a shrinking (boundedly complete) Schauder basis, then its $n$th conjugate space has a boundedly complete (shrinking) basis if $n$ is odd or a shrinking (boundedly complete) basis if $n$ is even. Furthermore if the first conjugate space of a quasi-reflexive space has a boundedly complete basis, then the space itself has a shrinking basis.

The last two sections give conditions under which a quasi-reflexive space is in fact reflexive: if every continuous linear functional attains its supremum on the unit sphere, or if the space is either smooth or rotund, then it is reflexive.

2. Definitions and notation. Let $B$ be a Banach space, $B^*$ and $B^{**}$ its first and second conjugate spaces. The canonical isomorphism of $B$ into $B^{**}$ will be denoted by $\pi$. If $A$ is a subspace of $B$ we will denote by $A^+$ the annihilator of $A$ in $B^*$. Following [1] we define $B$ to be quasi-reflexive of order $n$ if $B^{**}/\pi B$ is (finite) $n$-dimensional.

Let $w$ denote the set of positive integers. A sequence $(x_i; i \in w) \subset B$ will be called a basis if for each $x \in B$ there exists a unique sequence of real numbers $(a_i; i \in w)$ such that $\lim_{n \to \infty} \| \sum_{i=1}^n a_i x_i - x \| = 0$; we then write $x = \sum_{i \in w} a_i x_i$. A basis $(x_i; i \in w)$ for $B$ will be called boundedly complete if for each sequence $(a_i; i \in w)$ of real numbers such that the sequence $(\| \sum_{i=1}^n a_i x_i \|; n \in w)$ is bounded there exists an $x \in B$ such that $x = \sum_{i \in w} a_i x_i$; it will be called shrinking if $\lim_{n \to \infty} \| x^* \|_n = 0$ for each $x^* \in B^*$ where $\| x^* \|_n = \sup \{ x^*(x) : x = \sum_{i=1}^n a_i x_i \text{ and } \| x \| \leq 1 \}$, and monotone if for each $x = \sum_{i \in w} a_i x_i$ in $B$ $\sum_{i=1}^m a_i x_i$ is a non-decreasing function of $m$.

A Banach space is smooth if for each $x \in B$ with $\| x \| = 1$ there exists a unique $x^* \in B^*$ such that $\| x^* \| = 1$ and $x^*(x) = 1$. If for each $x^* \in B^*$ with $\| x^* \| = 1$ there exists at most one $x \in B$ such that $\| x \| = 1$ and $x^*(x) = 1$, then the space is called rotund. It is easy to see that this terminology is equivalent to that of [2]. Finally a Banach space is

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called sub reflexive [6] if the set of bounded linear functionals which attain their supremum on the unit sphere of $B$ is dense in $B^*$.

3. Basis in quasi-reflexive spaces. If $B$ is a quasi-reflexive space of order $n$ then, by definition, we write $B^{**} = \pi B \oplus F$ where $F$ is a finite $n$-dimensional subspace. It follows that if $(x_i; i \in w)$ is a basis for $B$ then the sequence $(y_1^{**}, \ldots, y_n^{**}, \pi x_i; i \in w)$, where $y_1^{**}, \ldots, y_n^{**}$ is a basis for $F$, is a basis for $B^{**}$ which we will call a basis corresponding to $(x_i; i \in w)$.

3.1. Theorem. Let $B$ be a quasi-reflexive space. If $B$ has a basis, then this basis is shrinking (boundedly complete) if and only if the corresponding basis in $B^{**}$ is shrinking (boundedly complete).

Proof. If $(x_i; i \in w)$ is a shrinking basis for $B$ and $(y_1^{**}, \ldots, y_n^{**}, \pi x_i; i \in w)$ a corresponding basis for $B^{**}$, then for $x^{***} \in B^{***}$ we have

$$\lim_{m} \|x^{***}\|_m = \lim_{m; m > n} \|x^{***}\|_m \leq \lim_{m} \sup_{\|x\| \leq 1} \{x^{***}(\pi x): x = \sum a_i x_i\}.$$ 

By Theorem 15 [3], $B^{***} = \pi_1 B^* \oplus (\pi B)^+$, where $\pi_1$ is the canonical isomorphism of $B^*$ into $B^{***}$, so if $x^{***} = \pi_1 x^* + y^{***}$, where $x^* \in B^*$ and $y^{***} \in (\pi B)^+$ we obtain

$$\lim_{m} \|x^{***}\|_m \leq \lim_{m} \sup_{\|x\| \leq 1} \{x^*(x): x = \sum a_i x_i\} = \lim_{m} \|x^*\|_m = 0.$$ 

The corresponding basis for $B^{**}$ is therefore shrinking.

If $(x_i; i \in w)$ is a boundedly complete basis for $B$, let $(a_i; i \in w)$ be a sequence of real numbers such that the sequence

$$\left(\sum_{i \in \text{min}(m, n)} a_i y_i^{**} + \sum_{n < i \in m} a_i \pi x_{i-n}; m \in w\right)$$

is bounded. Since $B$ is quasi-reflexive there exists a continuous projection of $B^{**}$ onto $\pi B$, the sequence $\{\sum_{n < i \in m} a_i \pi x_{i-n}; m \in w\}$ is bounded and there exists $x \in B$ such that $x = \sum_{i \in w} a_i \pi x_i$. Let $y^{**} = \sum a_i y_i^{**} + \pi x$, then $y^{**} = \sum a_i y_i^{**} + \pi x_{i-n}$ and the corresponding basis in $B^{**}$ is therefore boundedly complete. The converse in each case is clear.

3.2. Corollary. Let $B$ be a quasi-reflexive space with a monotone shrinking basis; then $B^{(n)}$ is sub reflexive for all $n \geq 0$ where $B^{(0)} = B$ and $B^{(n)} = (B^{(n-1)})^*$. 

Proof. This is a direct consequence of [6] and the above theorem.
3.3. Theorem. Let $B$ be a quasi-reflexive space. If $B$ has a boundedly complete basis then $B^*$ has a shrinking basis.

Proof. Let $(x_i; \ i \in \omega)$ be a boundedly complete basis for $B$. Let $(x_i^*; \ i \in \omega)$ be the sequence in $B^*$ biorthogonal to $(x_i; \ i \in \omega)$ and $A$ the closed subspace of $B^*$ spanned by $(x_i^*; \ i \in \omega)$. Then by Lemma 2 \cite[p. 70]{2}, $B^{**} = \pi B \oplus A^+$ and since $B$ is quasi-reflexive $A^+$ must be finite $n$-dimensional; let $y_i^**, \ i \leq n$ be a basis for $A^+$ and pick $y_i^*, \ i \leq n$ in $B^*$ such that $y_i^*(y_i^*) = \delta_{ij}$. Let $R$ be the subspace spanned by $y_1^*, \ \cdots, y_n^*, x_i^*; \ i \in \omega$ is a basis for $B^*$, since $(x_i^*; \ i \in \omega)$ is a basis for $A$.

Let $(x_i^**; \ i \in \omega)$ be the sequence in $B^{**}$ biorthogonal to $(y_i^*, \ \cdots, y_n^*, x_i^*, i \in \omega)$. We will show that $(x_i^**; \ i \in \omega)$ spans $B^{**}$; by Lemma 1 \cite[p. 70]{2} this will prove that the basis for $B^*$ is shrinking.

Let $(y_i^**, \ \cdots, y_n^**, y_i^*, i \in \omega)$ be the corresponding basis for $B^{**}$, then one verifies easily that $x_i^** = y_i^*$, $i \leq n$ and $x_n^* = \sum_{i \leq n} y_i^*(x_i)x_n^*$. If $x^** \in B^{**}$, $x^** = \pi x + y^*$ with $x \in B$ and $y^* = \sum_{i \leq n} \beta_i x_i^* \in A^+$ and one can write $x^** = \sum_{i \in \omega} x_i^*(x)x_i^* + \sum_{i \leq n} (\beta_i + y_i^*(x))x_i^*$.

Therefore the sequence $(x_i^**; \ i \in \omega)$ spans $B^{**}$ and the basis for $B^*$ is shrinking.

3.4. Corollary. Let $B$ be a quasi-reflexive space. If $B$ has a basis which is shrinking (boundedly complete) then there exist quasi-reflexive spaces $B(i), i \in \omega$ such that $B^0 = B$, $B^{(i+1)} = (B^{(i)})^*$, $B^{(2k)}$ has a shrinking (boundedly complete) basis and $B^{(2k+1)}$ has a boundedly complete (shrinking) basis.

Proof. The proof follows from Lemma 3.4 \cite{1}, Corollary 1 \cite[p. 71]{2} and Theorems 3.1 and 3.3 above.

Let $B^*$ have a basis. Whether or not this implies the existence of a basis for $B$ appears to be an open question. In case $B$ is quasi-reflexive the following theorem provides a partial answer.

3.5. Theorem. Let $B$ be a quasi-reflexive Banach space. If $B^*$ has a boundedly complete basis then $B$ has a shrinking basis.

Proof. Let $(x_i^*; \ i \in \omega)$ be a boundedly complete basis. By Lemma 3.4 \cite{1} $B^*$ is quasi-reflexive and so, in the proof of the preceding theorem $B^{***} = \pi_1 B^* \oplus A^+$ where $A$ is the subspace spanned by $(x_i^*; \ i \in \omega)$ the sequence biorthogonal to $(x_i^*; \ i \in \omega)$ and $A^+$ is finite $n$-dimensional. Also $B^{**} = R \oplus A$ where $R$ is finite $n$-dimensional and $A$ is a total subspace of $B^{**}$. By Theorems 14 and 16 of \cite{3} there exists an isomorphism $T$ from $B^*$ onto $A^*$ such that $(Tx^*)(y^*) = y^*(x^*)$ for all $x^* \in B^*$ and $y^* \in A^*$. By Theorem 3.6 of \cite{1} there exists an
isomorphism $S$ of $A$ onto $B$. It is easy to verify that the sequence $(Sx_i^*; i \in W)$ is a basis for $B$. To see that this basis is shrinking consider the sequence $(z_i^*; i \in W)$ in $B^*$ biorthogonal to $(Sx_i^*; i \in W)$. The sequence $(Tx_i^*; i \in W)$ in $A^*$ is biorthogonal to $(x_i^*; i \in W)$, therefore $Sy^* = S \sum_{i \in W} (Tx_i^*)(y^*)x_i^* = \sum_{i \in W} y^*(x_i^*)Sx_i^*$ for all $y^* \in A$; it follows that $z_i^*(Sy^*) = y^*(x_i^*)$ for all $i \in W$ and all $y^* \in A$; since $A$ is total we have $T^{-1}S^*z_i^* = x_i^*$ and so $(x_i^*; i \in W)$ is the isomorphic image of the basis $(x_i^*; i \in W)$. We conclude that $(Sx_i^*; i \in W)$ is a shrinking basis for $B$.

If $B$ is a reflexive space and $B$ has a basis, then it is well known (see, for example, [2]) that this basis must be both shrinking and boundedly complete. The following theorem holds for quasi-reflexive spaces.

3.6. Theorem. Let $B$ be a quasi-reflexive space of order 1, then any basis for $B$ is either shrinking or boundedly complete but not both.

Proof. Let $(x_i^*; i \in W)$ be a basis for the space $B$ which is quasi-reflexive of order 1. If $(x_i^*; i \in W)$ is not shrinking, then, by Lemma 1 [2, p. 70], $A$ the closed subspace spanned by $(x_i^*; i \in W)$, the sequence in $B^*$ biorthogonal to $(x_i^*; i \in W)$, is a proper subspace of $B^*$ and consequently $A^+$ is nonvoid. Since $B$ has order 1 we can write $B^* = \pi B \oplus \{ay^*\}$, $y^* \in \pi B$, $a$ any real number. Let $z_i^* \in A^+$; then $z_i^* = \pi x_0 + ay^*$ for some $x_0 \in B$ and $a_0 \neq 0$, and for any $x^* \in A$ we have $x^*(x_0) = -a_0y^*(x^*)$. If $z_i^* = \pi x + ay^* \in A^+$ we must have $x^*(x) = (a/a_0)x^*(x_0)$ for all $x^* \in A$; $A$ being total this implies that $x = (a/a_0)x_0$ and so $z_i^* = (a/a_0)(\pi x_0 + ay^*) = (a/a_0)z_i^*$. We conclude that $A^+ = \{az_i^*\}$ is 1-dimensional. It is easy to see that one can write $B^* = \pi B \oplus A^+$. By Lemma 2 [2, p. 70] this implies that $(x_i^*; i \in W)$ is boundedly complete. Finally if $(x_i^*; i \in W)$ were both shrinking and boundedly complete then $B$ would be reflexive by Theorem 3 [2, p. 71], contradicting the hypothesis.

4. The supremum of functionals in quasi-reflexive spaces. James [4] has shown that if all bounded linear functionals on a separable Banach space attain their supremum on the unit sphere then the space is reflexive. Whether this result is true for a general Banach space is not known. However the result can be extended to non-separable quasi-reflexive Banach space.

4.1. Theorem. Let $B$ be a quasi-reflexive Banach space. If all bounded linear functionals on $B$ attain their supremum on the unit sphere of $B$ then $B$ is reflexive.

Proof. Assume the hypothesis of the theorem. If $B$ is separable then
James' result can be applied. If $B$ is not separable then by Theorem 4.6 of [1] there exists a nonseparable reflexive subspace $Z$ such that $B/Z$ is separable.

The mapping $T$ of $Z^+$ onto $(B/Z)^*$ defined by $Tx^*(x+Z) = x^*(x)$ for all $x^* \in Z^+$ and $x \in B$, is an isometric isomorphism. If $x^* \in Z^+$ there exists $x_0 \in B$ such that $\|x_0\| = 1$ and $x^*(x_0) = \|x^*\|$, and therefore $Tx^*(x_0 + Z) = x^*(x_0) = \|x^*\| = \|Tx^*\|$. Every functional on $B/Z$ attains its supremum on the unit sphere of $B/Z$ as well as $Z$ is reflexive. It follows that $B$ is reflexive.

5. Rotundity and smoothness in quasi-reflexive spaces.

5.1. Theorem. If $B$ is a quasi-reflexive Banach space, there exists an equivalent norm for $B$ such that if $B$ is either rotund or smooth in this norm, then $B$ is reflexive.

Proof. By Theorem 3.5 [1] there exists an equivalent norm for $B$ and quasi-reflexive Banach spaces $B^{(-4)}$, $B^{(-4)}$ and $B^{(-1)}$ such that $(B^{(-4)})^{****} = B^{(-1)}$, $(B^{(-4)})^{****} = B$ and $(B^{(-1)})^* = B$. By Theorem 20 [3], if $B$ is rotund in this norm we must have $B^{(-4)}$ reflexive. It follows that $B$ is reflexive. If $B$ is smooth, then by [5, p. 12] $B^{(-1)}$ is rotund and so $B^{(-4)}$ is reflexive; $B$ is therefore reflexive.

Bibliography


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