APPROXIMATION BY BOUNDED ANALYTIC FUNCTIONS TO FUNCTIONS REPRESENTED BY DIRICHLET SERIES

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Results concerning approximation to functions analytic on a closed point set $\overline{R}_0$ by arbitrary functions analytic and bounded in a region R_1 containing $\overline{R}_0$ were first established by Walsh [4] in 1938 and later extended by the present writers to the limiting case where $\overline{R}_0$ and the boundary of R_1 have points in common [5]. It is the purpose of the present note to continue the study of this problem now in situations where the approximated function is no longer assumed analytic at points common to the boundaries of R_0 and R_1.

Let the respective boundaries C_0 and C_1 of R_0 and R_1 be Jordan curves intersecting in the single point A. We suppose also that R_1 is mapped one-to-one and conformally by $w=\theta(z)$, $z=\chi(w)$ onto $\operatorname{Re} w>u_1$ and that this mapping carries C_0 onto $\operatorname{Re} w=u_0$, $u_1<0<u_0$. The function $f(z)$ will be said to belong to class $\mathfrak{D}$ if $g(w)=f(\chi(w))$ can be represented by a Dirichlet series

$$(1) \qquad \sum_{\nu=1}^{\infty} a_\nu e^{-\lambda_\nu w}, \qquad 0 \leqq \lambda_1 < \cdots < \lambda_\nu \cdots, \qquad \lim_{n\to\infty} \lambda_\nu = \infty,$$

converging to $g(w)$ throughout every half plane of analyticity of $g(w)$, uniformly throughout every half plane contained in the half plane of convergence.

Our main result is the

THEOREM. *Let the Jordan curves C_0 and C_1 satisfy the conditions stated above. Let $\phi(z)$ be harmonic and bounded in $R_1-\overline{R}_0$, continuous in $\overline{R}_1-R_0-A$, and equal to zero and unity on C_0-A and C_1-A respectively. Denote generically by C_σ the locus $\phi(z)=\sigma$, $0<\sigma<1$, in $R_1-\overline{R}_0$ and by R_σ the point set consisting of $\overline{R}_0-A$ plus those points of $\overline{R}_1-R_0$ for which $0<\phi(z)<\sigma$. Let the function $f(z)$ of class $\mathfrak{D}$ be analytic throughout some R_ρ, $0<\rho<1$, but not analytic throughout any $R_{\rho'}$, $\rho<\rho'$.*

Let $f_M(z)$ be the (or a) function analytic and of modulus not greater than M in R_1 for which l.u.b. $[|f(z)-f_M(z)|$, *z in $\overline{R}_0-A$] is least. Then* $\lim_{M\to\infty} f_M(z)=f(z)$ *throughout R_ρ, uniformly on any closed subset of R_ρ and we have*

Received by the editors November 23, 1960.

[1] With respect to the second named author, this research was sponsored (in part) by the U. S. Air Force, Air Research and Development Command.

$$(2)\quad \limsup_{M\to\infty} [\text{l.u.b.}\,|f(z)-f_M(z)|\,,\ z \text{ in } \overline{R}_\sigma - A]^{1/\log M} = e^{(\sigma-\rho)/(1-\rho)},\qquad 0 \leqq \sigma < \rho,$$

$$(3)\quad \limsup_{M\to\infty} [\text{l.u.b.}\,|f_M(z)|\,,\ z \text{ in } R_\sigma]^{1/\log M} = e^{(\sigma-\rho)/(1-\rho)},\qquad \rho \leqq \sigma \leqq 1.$$

Equalities (2) and (3) are known to hold when the approximated function (not of class $\mathfrak{D}$) is analytic at A. In that event the extremal functions corresponding to $f_M(z)$ for the case where C_0 and C_1 have no points in common may be employed as comparison functions to establish (2) and (3) since $f(z)$ can be extended analytically across C_0 (see [5]). Such a method of attack is no longer applicable to the situation of this note. Instead, suitable partial sums of series (1) will be used for comparison purposes in the proof of the present theorem. Thus, Dirichlet series now enter for the first time the general study of approximation by bounded analytic functions.

Since the content of the theorem is invariant under an arbitrary one-to-one conformal map, the situation in the z-plane may be transferred to the w-plane by the mapping $w=\theta(z)$, $z=\chi(w)$. Set $g(w)=f(\chi(w))$, $g_M(w)=f_M(\chi(w))$ and let $\operatorname{Re} w=u_\sigma$ denote generically the image in the w-plane of C_σ, $0<\sigma<1$. We may assume $u_\rho=0$, in which case $(\sigma-\rho)/(1-\rho)=u_\sigma/u_1$. Equalities (2) and (3) take the following forms:

$$(2)\quad \limsup_{M\to\infty} [\text{l.u.b.}\,|g(w)-g_M(w)|\,,\ \operatorname{Re} w \geqq u_\sigma]^{1/\log M} = e^{u_\sigma/u_1},\qquad 0 < u_\sigma \leqq u_0,$$

$$(3)\quad \limsup_{M\to\infty} [\text{l.u.b.}\,|g_M(w)|\,,\ \operatorname{Re} w > u_\sigma]^{1/\log M} = e^{u_\sigma/u_1},\qquad u_1 \leqq u_\sigma \leqq 0.$$

We state now two properties of series (1) upon which the proof of the theorem rests [1, pp. 8–9]. Let

$$g_n(w) = \sum_{\nu=1}^{n} a_\nu e^{-\lambda_\nu w}.$$

For given $u_\sigma' < u_\sigma$ and for all n we have

$$(4)\qquad |g(w)-g_n(w)| \leqq Ke^{-u_\sigma'\lambda_{n+1}},\qquad \operatorname{Re} w \geqq u_\sigma,\qquad 0 < u_\sigma \leqq u_0,$$

$$(5)\qquad |g_n(w)| \leqq K'e^{-u_\sigma'\lambda_n},\qquad \operatorname{Re} w > u_\sigma,\qquad u_1 \leqq u_\sigma \leqq 0.$$

Here and below, K and K' with or without subscripts denote constants independent of w and n (or M).

We proceed to establish (2) and (3) in their alternate forms. Let the numbers u_0', u_1' satisfy $u_1' < u_1 < 0 < u_0' < u_0$. Let K_1' be the value of K' in (5) corresponding to $u_\sigma = u_1$. For given $M > K_1' e^{-u_1'\lambda_1}$ choose n so that

$$K_1' e^{-u_1'\lambda_n} \leqq M < K_1' e^{-u_1'\lambda_{n+1}}. \tag{6}$$

For this choice of n, $|g_n(w)| \leqq M$, $\operatorname{Re} w > u_1$. The function $g_M(w)$ is a function analytic and of modulus not greater than M in $\operatorname{Re} w > u_1$ for which the quantity l.u.b. $[|g(w) - g_M(w)|, \operatorname{Re} w \geqq u_0]$ is least. It then follows from (4), with $u_\sigma = u_0$, $K = K_1$, and from (6) that

$$\begin{aligned} &\text{l.u.b.}[|g(w) - g_M(w)|, \operatorname{Re} w \geqq u_0] \\ &\qquad \leqq \text{l.u.b.}[|g(w) - g_n(w)|, \operatorname{Re} w \geqq u_0] \\ &\qquad \leqq K_1 e^{-u_0'\lambda_{n+1}} < K_1 \exp\{u_0'(\log M - \log K_1')/u_1'\} \\ &\qquad = K_2 \exp\{(u_0' \log M)/u_1'\}. \end{aligned} \tag{7}$$

Thus, the first member of (2) cannot exceed the second member when $u_\sigma = u_0$ since u_0' and u_1' can be taken arbitrarily close to the respective quantities u_0 and u_1.

Now let u_σ, u_σ' be given with $0 < u_\sigma' < u_\sigma < u_0$. Choose u_0', u_1', $u_1' < u_1 < 0 < u_\sigma < u_0' < u_0$, so that $u_1'(u_0 - u_\sigma) + u_0'(u_\sigma - u_1) - u_\sigma'(u_0 - u_1) \geqq 0$. Choice of u_0' and u_1' is possible since the left-hand member of this last inequality is a continuous function of u_0' and u_1' and is positive when $u_0' = u_0$, $u_1' = u_1$. As before let n be chosen so that (6) is satisfied. Then

$$\begin{aligned} &|g_n(w) - g_M(w)| \leqq 2M, \qquad \operatorname{Re} w > u_1, \text{ and from (7)} \\ &|g_n(w) - g_M(w)| < 2K_2 \exp\{(u_0' \log M)/u_1'\}, \qquad \operatorname{Re} w \geqq u_0. \end{aligned}$$

It now follows, by virtue of Doetsch's three-line-theorem [2] and choice of u_0', u_1', that

$$\begin{aligned} &\text{l.u.b.}[|g_n(w) - g_M(w)|, \operatorname{Re} w = u_\sigma] \\ &\qquad \leqq [2M]^{(u_0 - u_\sigma)/(u_0 - u_1)} [2K_2 \exp\{u_0' \log M/u_1'\}]^{(u_\sigma - u_1)/(u_0 - u_1)} \\ &\qquad \leqq K_2' \exp\{u_\sigma' \log M/u_1'\}. \end{aligned} \tag{8}$$

Employing (4) once more for given u_σ, and with corresponding $K = K_3$, we have because of (6),

$$\begin{aligned} |g(w) - g_n(w)| &\leqq K_3 e^{-u_\sigma'\lambda_{n+1}} < K_3 \exp\{u_\sigma'(\log M - \log K_1')/u_1'\} \\ &= K_3' e^{u_\sigma' \log M/u_1'}, \qquad \operatorname{Re} w \geqq u_\sigma. \end{aligned}$$

This last inequality, the inequality in (8) and an extension of the maximum principle for subharmonic functions [5], now imply

$$\begin{aligned} &\text{l.u.b.}[|g(w) - g_M(w)|, \operatorname{Re} w \geqq u_\sigma] \\ &\qquad \leqq 2 \max(K_2', K_3') \exp\{(u_\sigma' \log M)/u_1'\}. \end{aligned}$$

Since u_σ' and u_1' can be taken arbitrarily close to u_σ and u_1 respec-

tively, the first member of (2) cannot exceed the second member.

That the left member of (3) cannot be greater than the right for $u_\sigma = u_1$ is an immediate consequence of the inequality $|g_M(w)| \leqq M$, $\operatorname{Re} w > u_1$. Similar inequalities in (3) for general u_σ, $u_1 < u_\sigma \leqq 0$, follow from (5), (6), (8) by the same methods as those used above to establish the corresponding inequalities in (2).

Equality in (2) and (3) is a consequence of the three-line-theorem and the definition of ρ. The details of proof appear elsewhere [**4**; **5**]. Convergence of $g_M(w)$ to $g(w)$ follows from (2).

A number of corollaries to the theorem are readily established. It may be noted that (3) is a consequence of (2) when (2) is known to hold for the analytic functions $f_M(z)$, not necessarily extremal but satisfying $|f_M(z)| \leqq M$ in R_1; the proof indicated above remains valid. When the extremal functions $f_M(z)$ in the left hand member of (2) are replaced by functions $F_M(z)$ analytic and of modulus no greater than M in R_1, but *otherwise arbitrary*, the expression thus obtained cannot be less than the right hand member of (2) (see [**3**]).

If $\{M_n\}$ *is an increasing sequence of positive numbers for which*

$$\lim_{n\to\infty} [\log M_{n+1}/\log M_n] = 1, \qquad \lim_{n\to\infty} M_n = \infty, \tag{9}$$

equality still holds in (2) *and* (3) *for this sequence and the corresponding extremal functions* $f_{M_n}(z)$. Condition (9) is more general than similar conditions which have been previously established in the literature. We give the proof for equality in (2). Then (3) follows from (2) as already noted. Let M be an arbitrary positive number, $M \geqq M_1$. We set $F_M(z) \equiv f_{M_n}(z)$, $M_n \leqq M < M_{n+1}$, where, of course, $M_n = M_n(M)$. Then for sufficiently large M we have

$$0 < \log M_n \leqq \log M < \log M_{n+1},$$

$$1 \leqq \log M/\log M_n < \log M_{n+1}/\log M_n, \qquad M_n \leqq M < M_{n+1}.$$

Hence, by virtue of (9)

$$\lim_{M\to\infty} [\log M/\log M_n] = 1,$$

$$\begin{aligned}\limsup_{M_n\to\infty} &[\text{l.u.b.}\ |f(z) - f_{M_n}(z)|,\ z \text{ in } \overline{R}_\sigma - A]^{1/\log M_n} \\ &= \limsup_{M\to\infty} [\text{l.u.b.}\ |f(z) - F_M(z)|,\ z \text{ in } \overline{R}_\sigma - A]^{1/\log M} \\ &\leqq e^{(\sigma-\rho)/(1-\rho)}.\end{aligned}$$

The strong inequality would contradict the remark made in the preceding paragraph concerning the $F_M(z)$.

Results analogous to those in (2) and (3) can be obtained by the same methods used in this paper when we study the companion problem of approximation to $f(z)$ by functions $f_m(z)$ of minimum norm. Here, for given m (>0), $f_m(z)$ is the (or a) function analytic in R_1, with $|f(z)-f_m(z)| \leqq m$ in $\overline{R}_0-A$, for which l.u.b. $[|f_m(z)|, z \text{ in } R_1]$ is least; [compare **4**; **5**].

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