CLASSES OF $p$-VALENT STARLIKE FUNCTIONS

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1. Introduction. The winding number associated with a starlike function exhibits a certain monotonicity property (Theorem 1 below). This property is used to show that several alternatives to the definition of the class $S(p)$ of $p$-valent starlike functions are trivial. From it there also follows a simple and explicit example of a coefficient problem in $S(p)$ with no solution. This situation, which Goodman has treated in some detail [2], is interesting since such problems always have solutions in the schlicht case [3; 4].

Let $S$ be the class of all functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ which are regular and schlicht in $|z| < 1$ and let $S^*$ be the subclass of $S$ consisting of those functions whose image domains are starshaped with respect to the origin. For a given positive integer $p$ let $S(p)$, the class of $p$-valent starlike functions, be the class of all functions $f$ to which there corresponds some $r$, $0 < r < 1$, such that for any $z$, $r < |z| < 1$, $\text{Re}\{zf'(z)/f(z)\} \geq 0$ and $(1/2\pi)\int_0^\pi \text{Re}\{zf'(z)/f(z)\} \, dt = p$, $z = qe^{it}$, for each $q$, $r < q < 1$. This integral is just the number of zeros of $f$ in the interior of the circle $|z| = q$ and hence $f$ has $p$ zeros in the open unit disk, and is in fact $p$-valent there [2]. In a certain sense the classes $S(1)$ and $S^*$ coincide, i.e., if $f \in S^*$ then $f \in S(1)$ and if $f \in S(1)$ then $f/f'(0) \in S^*$.

2. The winding number. If $Q$ is a path in $U = \{z: |z| < 1\}$ and $f$ is analytic in $U$, let $f(Q)$ denote the path which is the image of $Q$ under $f$ and which has the induced orientation. The properties of the winding number

$$n[f(Q), a] = \frac{1}{2\pi i} \oint_Q \frac{f'(z)}{f(z) - a} \, dz$$

are well known. In this paper $f$ will lack singularities and $Q$ will be a circle. Hence $n[f(Q), a]$ will be the number of times $f$ assumes the value $a$ in the open disk bounded by $Q$.

If a function $f$ is regular at a point $a \neq 0$ and $\text{Re}\{zf'(z)/f(z)\} \geq 0$ for all $z$ in some neighborhood $M$ of $a$ then the fact that $\text{Re}\{zf'(z)/f(z)\}$ is harmonic in some neighborhood $N \subset M$ of $a$ implies that $\text{Re}\{af'(a)/f(a)\} > 0$ and, consequently, also that $f'(a) \neq 0$. From this fact, which will be of frequent use below, follows

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Theorem 1. Let the function \( f \) be regular in the open unit disk \( U \) and zero at the origin. Suppose there is an \( r = r(f) \), \( 0 < r < 1 \), such that 
(i) \( f(z) \neq 0 \) and 
(ii) \( \text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} \geq 0 \), whenever \( r < |z| < 1 \). Then for any \( q \), \( r < q < 1 \), and associated \( Q = \{ z : |z| = q \} \) and any \( a \in U \), the winding number \( n[f(Q), sf(a)] \) is a decreasing function of the positive real variable \( s \) as long as \( sf(a) \in f(U) \).

Proof. Let \( f(U) \) be the image of \( U \) under \( f \). The winding number \( n[f(Q), sf(a)] \) is a nonnegative integer (hence real) and is constant throughout each component of \( f(U) \) determined by \( f(q) \). Therefore, as has just been noted, \( \text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} = \partial \text{arg} \, f(z)/\partial \text{arg} \, z > 0 \) whenever \( r < |z| < 1 \), i.e. \( \text{arg} \, f \) is a strictly increasing function of \( \text{arg} \, z \) for \( z \in Q \). Furthermore the fact that \( f \) is never zero in \( \{ z : r < |z| < 1 \} \) and has only a finite number \( m > 0 \) of zeros in \( \{ z : |z| \leq r \} \) implies, with the help of the argument principle, that \( \text{arg} \, f(z) \) increases by \( 2m\pi \) as \( z \) makes one positively directed circuit of \( Q \). Thus \( \text{Arg} \, f(z) \) takes on each value \( b, 0 \leq b < 2\pi \) precisely \( m \) times as \( z \) traverses \( Q \).

If \( a \in Q \) is arbitrary it is apparent that the angle \( \theta \) from the radius vector \( f(a) - 0 \) to the vector tangent to \( f(Q) \) at \( f(a) \) lies in the interval \( 0 < \theta < \pi \). The geometric meaning of the winding number now makes it obvious that its value falls as \( f(Q) \) is crossed in an outward direction and in fact that this decrease is just some integer \( n, 1 \leq n \leq m \), which is the number of points of \( Q \) mapped into \( f(a) \) by \( f \). The proof of the theorem is now complete.

3. The class \( M^*_p \).

Definition 1. Let \( p \) be fixed, \( p \geq 1 \). Suppose the function \( h(z) = z^p + b_{p+1}z^{p+1} + b_{p+2}z^{p+2} + \cdots \) is regular in \( U \) and satisfies the conditions

(i) \( h \) is \( p \)-valent in \( U \), and

(ii) there exists \( r = r(h) \), \( 0 < r < 1 \), such that \( \text{Re}\left\{ zh'(z)/h(z) \right\} \geq 0 \) whenever \( r < |z| < 1 \).

Then \( h \) is called a function of class \( M^*_p \).

Manifestly if \( h \in M^*_p \) then for any \( q, r < q < 1 \), and \( Q = \{ z : |z| = q \} \) it must be that \( n[h(Q), 0] = p \), i.e., \( (1/2\pi) \int_{0}^{2\pi} \text{Re}\left\{ zh'(z)/h(z) \right\} dt = p \), \( z = qe^{it} \), so that \( h \in S(p) \). Thus \( M^*_p \subset S(p) \). Let \( (S^*_p) \) be the class of \( p \)th powers of functions of \( S^* \). Then

Theorem 2. \( M^*_p = (S^*_p)^p, \quad p = 1, 2, 3, \cdots \).

Proof. Choose any function

\[
h(z) = z^p + b_{p+1}z^{p+1} + b_{p+2}z^{p+2} + \cdots
\]

\[
= z^p(1 + b_{p+1}z + b_{p+2}z^2 + \cdots) = z^pg(z)
\]
of $M_p^*$. Note that $g$ is regular and nonzero in $U$. Hence so is $[g(z)]^{1/p}$. Since $g(0) = 1$ the function $[g(z)]^{1/p}$ can be assumed to be $1$ at the origin. Set $f(z) = z[g(z)]^{1/p} = z + \cdots$. Then $[f(z)]^p = h(z)$. Furthermore, if $z \in U$, $zh'(z)/h(z) = p(f'(z)/f(z))$ whence, since $p > 0$, $\text{Re}\{zh'(z)/h(z)\} \geq 0$ whenever $r(h) = r < |z| < 1$. Hence, just as above, $\arg f(z)$ is a strictly increasing function of $\arg z$ for $z \in Q$. Therefore $f(Q)$ is a simple closed curve and by Darboux’s Theorem $f$ is schlicht. Thus $f \in S^*$, and consequently $M_p^* \subset (S^*)^p$.

If, now, $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ is a member of $S^*$ and the function $h(z) = z^p + \cdots$ is defined by setting $h(z) = [f(z)]^p$ then certainly $h \in M_p^*$, for $f$ can take on each $p$th root of a given number at most once in $U$, showing that $h$ satisfies condition (i). That $h$ satisfies condition (ii) has already been shown in the first half of this proof. Therefore $(S^*)^p \subset M_p^*$ and the theorem is proved.

4. The class $N_p^*$.

Definition 2. Let $p$ be fixed, $p \geq 1$. Let $m$ be an integer, $1 \leq m \leq p$. Suppose the function $h(z) = z^m + b_{m+1}z^{m+1} + b_{m+2}z^{m+2} + \cdots$ is regular in $U$ and satisfies the conditions

(i) $h$ is at most $p$-valent in $U$, and

(ii) $\text{Re}\{zh'(z)/h(z)\} \geq 0$ for all $z \in U$.

Then $h$ is called a function of class $N_p^*$.

The relationship of the class $N_p^*$ to $S(p)$ is made plain by the following

**Theorem 3.** $N_p^* = S^* \cup (S^*)^2 \cup \cdots \cup (S^*)^p$, $p = 1, 2, 3, \cdots$.

**Proof.** Consider an arbitrary $h \in N_p^*$. The function $h$ is of the form $h(z) = z^m + b_{m+1}z^{m+1} + b_{m+2}z^{m+2} + \cdots$ for some integer $m$, $1 \leq m \leq p$. To show that $h \in M_m^* = (S^*)^m$ it clearly suffices to verify that $h$ is $m$-valent in $U$. Consider any point $a \neq 0$ of $U$. If $h(a) = 0$ then, in some neighborhood of $a$, $h(z) = (z - a)^n g(z)$ where $1 \leq n$ and $g(a) \neq 0$ (for the identically zero function is not a member of $N_p^*$). Then $zh'(z)/h(z) = na/(z - a) + n + zg'(z)/g(z)$. Certainly $n + zg'(z)/g(z)$ is analytic at $z = a$ since $g(a) \neq 0$ and therefore $zh'(z)/h(z)$ has a pole of order $1$ at $z = a$ in contradiction to condition (ii) in the hypothesis concerning $h$. Thus $h$ can be zero in $U$ only at the origin. Hence for any $q$, $0 < q < 1$, and associated $Q = \{z: |z| = q\}$ and $G = \{z: |z| < q\}$ it is apparent that $n[h(Q), h(0)] = n[h(Q), 0] = m$. Theorem 1 now guarantees that $h \in M_m^* \subset (S^*)^m \subset (S^*)^\infty$ whenever $z \in G$. Since $q$ can be arbitrarily close to 1, $h$ must be $m$-valent in $U$. Thus $N_p^* \subset S^* \cup (S^*)^2 \cup \cdots \cup (S^*)^p$.

For $m = 1, 2, 3, \cdots$, $p$ the proof that $(S^*)^m \subset N_p^*$ is the same as
5. The class $S^*_p$.

**Definition 3.** Let $p = 1, 2, 3, \ldots$. Suppose the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ is regular in $U$ and satisfies the conditions

(i) $f$ is at most $p$-valent in $U$,
(ii) there exists some $a \in \mathcal{U}(U)$ such that $f(z) = a$ exactly $p$ times in $U$, and
(iii) there exists $r = r(f)$, $0 < r < 1$, such that $\Re\left\{z f'(z)/f(z)\right\} \geq 0$ whenever $r < |z| < 1$.

Then $f$ is called a function of class $S^*_p$.

Evidently $S^*_1 = S^*$ which, in turn, is in the sense above alluded to just equal to $S(1)$. But not even in this sense is $S^*_p = S(p)$. However

**Theorem 4.** $S^*_p \subset S(p)$, $p = 1, 2, 3, \ldots$.

**Proof.** If $f \in S^*_p$ then all that must be verified to show that $f \in S(p)$ is the existence of a $d$, $0 < d < 1$, such that whenever $d < q < 1$

$$n\left[f(Q), 0\right] = \frac{1}{2\pi} \int_0^{2\pi} \Re\left\{\frac{zf'(z)}{f(z)}\right\} \, dt = p,$$

where

$$z = q e^{it}.$$

There are some $a \in \mathcal{U}$, some number $k$, $0 < k < 1$, and a circle $K = \{z: |z| = k\}$ contained in $U$ such that $n\left[f(K), f(a)\right] = p$. Let $c$, $0 < c < 1$, be the largest of the moduli of the (at most $p$) points $z_j$ of $U$ at which $f(z_j) = 0$. The number $r = r(f)$ is already associated with $f$. Define $d = \max\{r, k, c\}$. Then for any $q$, $d < q < 1$, and associated $Q = \{z: |z| = q\}$ it is true that $n\left[f(Q), f(a)\right] \leq p$. The fact that $f$ is at most $p$-valent in $U$ implies that $n\left[f(Q), 0\right] \leq p$. Hence an application of Theorem 1 to $f$ yields $\hat{p} = n\left[f(Q), f(a)\right] \leq n\left[f(Q), 0\right] \leq p$. Since $q$ can be arbitrarily close to 1 this shows that $f \in S(p)$, i.e. $S^*_p \subset S(p)$, which completes the proof.

If $g$ is any function of $S(p)$ having a simple zero at the origin then $g$ can be written $g(z) = a_2z + a_3z^3 + \cdots$. It is now a trivial matter to verify that

$$g(z) = g\left(z + \frac{a_2}{a_1}z^2 + \frac{a_3}{a_1}z^3 + \cdots\right)$$

is an element of $S^*_p$. And in this sense, the same as with the classes $S(1)$ and $S^* = S^*_1$ (i.e. except for normalization) the class $S^*_p$ is just the class of all functions of $S(p)$ having a simple zero at the origin.
It is clear that for $p = 2, 3, 4, \ldots$ no function of $S^*_p$ is the power of a (schlicht) starlike function.

The following theorems give an example of a coefficient problem which has no solution.

**Theorem 5.** Let $p$ be fixed, $p \geq 2$. If $k$ is any complex number such that $2 - 1/p < |k|$ then the function $f(z) = z + k z^p$ is a member of $S^*_p$.

**Proof.** The function $f$ has $p$ zeros in $U$ and is obviously $p$-valent in $U$. Also, for $|z| = 1$,

$$\text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} = \text{Re} \left\{ p + \frac{(1 - p)z}{z + k z^p} \right\} \geq p - \frac{p - 1}{|k| - 1} \left( \frac{1}{p} \right) > 0,$$

which persists, by continuity, for $r < |z| < 1$, for some $r$, $0 < r < 1$. Therefore $f \in S^*_p$.

**Theorem 6.** Let $p$ be fixed, $p \geq 2$. Let $n \neq p$ be chosen, $n \geq 2$. Then to any complex number $q$ there corresponds a function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ of $S^*_p$ for which $a_n = q$.

**Proof.** If $q = 0$ the theorem is an immediate consequence of Theorem 5. So consider arbitrary fixed nonzero $q$, and let $b$ be any complex number whose modulus satisfies the inequality $1 + |q| < |p| + |n - p| \cdot |q| < |b|$. Now consider the function $f(z) = z + b z^p + q z^n$. In consequence of the above inequality and a theorem of Pellet [1, p. 10] on roots of polynomials it follows immediately that $n[f(C), 0] = p$, where $C = \{z: |z| = 1\}$. The fact that $\max_{z \in C} |zf''(z)/f'(z) - p| < 1 < p$ implies, just as in the proof of Theorem 5, that there exists $d$, $0 < d < 1$, such that $\text{Re} \{zf''(z)/f(z)\} \geq 0$ whenever $d < |z| < 1$. Since $f$ takes on only $p$ zeros in $U$ there exists $t$, $0 < t < 1$, such that $f(z) \neq 0$ whenever $t < |z| < 1$. Set $r = \max(d, t)$. Then $0 < r < 1$ and Theorem 1 is applicable to $f$. Hence if $r < x < 1$ and $X = \{z: |z| = x\}$ the winding number $n[f(X), sf(a)]$ is a decreasing function of the positive real variable $s$ whenever $sf(a) \in f(U)$. Thus $p = n[f(C), 0] = n[f(X), 0] \geq n[f(X), k]$ for any $k \in f(U)$. But $x$ can be arbitrarily close to 1. Therefore $f \in S^*_p$ and the theorem is proved.

Consideration of the classes of $p$-valent starlike functions treated above has given rise to the following question concerning a decomposition for elements of $S(p)$. Given any $f \in S(p)$, does $f$ have a representation $f = gh$ where $g \in S^*_p$, $h \in (S^*)^{p-1}$?
REFERENCES


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