THE POWER OF TOPOLOGICAL TYPES OF SOME CLASSES OF 0-DIMENSIONAL SETS

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By a result of Mazurkiewicz and Sierpinski, there exist $\aleph_1$ topological types of compact and countable sets. Since a countable set is 0-dimensional, there arises a natural question: what is the power of topological types of other classes of 0-dimensional sets? In this paper we consider separable metric spaces only. Every 0-dimensional space being topologically contained in the Cantor set $C$, we confine ourselves to subsets of this set.

We prove the following three theorems:

**Theorem 1.** There exist two topological types of open subsets of the Cantor set $C$.

**Theorem 2.** There exist $2^{\aleph_0}$ topological types of closed subsets of the Cantor set $C$.

**Theorem 3.** There exist $2^{\aleph_0}$ topological types of 0-dimensional $G_\delta$ sets which are dense in themselves.

Theorem 1 is known in part, but it seems to the author that an exact proof of it has not been published so far.

Theorems 2 and 3 are new; the latter gives an answer to a problem by Knaster and Urbanik.

The paper contains also some lemmas on homeomorphisms and a notion of a rank $r_p(B)$ of a point $p$ relative to the set $B$.

1. In this section a lemma on homeomorphisms and the above Theorem 1 are proved.

**Lemma 1.** Let $\{F_n\}$ and $\{G_n\}$ be two sequences of sets satisfying

1. $F_n \cap F_m = 0 = G_n \cap G_m$ for $n \neq m$,
2. for every $n$ the set $F_n$ is open in the union $F = \bigcup_{n=1}^{\infty} F_n$ and $G_n$ is open in $G = \bigcup_{n=1}^{\infty} G_n$, and
3. for every $n$ there exists a homeomorphism $h_n$ such that $h_n(F_n) = G_n$, $n = 1, 2, \ldots$.

Then the mapping $h$ defined by $h(x) = h_n(x)$ for $x \in F_n$ is a homeomorphism between $F$ and $G$.

Received by the editors July 21, 1960 and, in revised form, December 6, 1960.

1 See [6, p. 22].
2 See [4, p. 173].
3 Some general hints may be found in [3, p. 198].
4 See [3, p. 198].
PROOF. By (1) and (3), \( h \) is a one-to-one mapping of \( F \) onto \( G \). Since the proofs of the continuity of \( h \) and \( h^{-1} \) are symmetric, we shall show that \( h \) is continuous.

Indeed, let \( \{x_n\} \) be a sequence of points belonging to \( F \), tending to a point \( x \) of \( F \); \( x_n\to x \). Since \( x\in F \), there exists a number \( n_0 \) such that \( x\in F_{n_0} \). Now by (2) there exists a number \( N \) such that for \( n>N \) there is \( x_n\in F_{n_0} \) (since otherwise the set \( F_{n_0} \) would not be open in \( F \)). But \( h_{n_0} \) is continuous—as a homeomorphism—and therefore for \( n>N \):

\[
    h(x_n) = h_{n_0}(x_n) = h_{n_0}(x) = h(x).
\]

Remark 1. Let \( F_n \) be the plane set defined by \( F_n = \{(x, y); x = 1/n, 0 \leq y \leq 1\} \) and put \( G_n = \{(x, y); x = 0; 0 \leq y \leq 1\} \) and \( G_{n+1} = F_n \), \( n = 1, 2, \ldots \). For these sets the assumption (2) of the lemma is not satisfied for the set \( G_1 \) only and evidently \( F = \bigcup_{n=1}^\infty F_n \) is not homeomorphic with \( G = \bigcup_{n=1}^\infty G_n \), since \( G \) is a compact set and \( F \) is not. This shows also that assumption (2) of the lemma cannot be replaced by the assumption that \( F_n \) and \( G_n \) are compact and \( \rho(F_n, F_m) \) and \( \rho(G_n, G_m) \) are positive for all \( n \neq m \).

To prove Theorem 1 it suffices to show that:

Every open subset of the Cantor set \( C \) is either homeomorphic to \( C \) or to \( C \) without the zero point: \( C \setminus \{0\} \).

PROOF. Let \( G \) be an open subset of the Cantor set \( C \). Then \( G \) can be written in the form:

\[
    G = G_1 \cup G_2 \cup \cdots, \quad G_n \cap G_m = 0 \quad \text{for} \quad n \neq m,
\]

where the sets \( G_n \) are closed and open in \( C \).

Now two cases are possible:

(a) \( G \) is a finite union of the sets \( G_n \), i.e. there exists an integer \( N \) such that \( G_n = 0 \) for \( n > N \), and

(b) all the sets \( G_n \) in (4) are nonempty.

Since

(5) a closed and open subset of the Cantor set \( C \) is a perfect set, we see that in case (a) the set \( G \) is a perfect 0-dimensional set and therefore homeomorphic to the Cantor set \( C \).

In case (b) we can write the set \( C \setminus \{0\} \) analogically as in (4) in the form:

\[
    C \setminus \{0\} = F_1 \cup F_2 \cup \cdots, \quad F_n \cap F_m = 0 \quad \text{for} \quad n \neq m,
\]

where the sets \( F_n \) are nonempty and closed and open in \( C \).

By (5) there exists for every \( n \) a homeomorphism \( h_n \) between \( F_n \) and \( G_n \) and therefore by (4) and (6) the assumptions of the lemma hold.

\footnote{By \( \rho(F_n, F_m) \) we understand the distance between the sets \( F_n \) and \( F_m \), i.e. \( \rho(F_n, F_m) = \inf_{x \in F_n, y \in F_m} \rho(x, y) \), where \( \rho(x, y) \) denotes the distance between the points \( x \) and \( y \).}

\footnote{See [4, p. 166].}
Thus by the lemma the set $G$ is, in case (b), homeomorphic to $C \setminus \{0\}$.

Remark 2. Theorem 1 may also be proved in another way by using the one-point compactification theorem, but such an exact proof is not simpler than ours.

2. We show in this section that there exist $2^{|S|}$ topological types of closed subsets of the Cantor set $C$. Since the power of all closed subsets of $C$ is $2^{|S|}$ and every 0-dimensional space has a topological image in the Cantor set $C$, it suffices to construct a family $S$ of power $2^{|S|}$ of compact, 0-dimensional sets, such that no two sets belonging to this family are homeomorphic. To do this we introduce the notion of a rank $r_\alpha(B)$ of a point $p$ relative to the set $B$. First we recall the notion of the coherence and adherence of a set $E$ in the sense of Hausdorff.

Evidently, the $\alpha$th adherence is an isolated set. The $\alpha$th adherence of the set $E$ will be denoted by $E(\alpha)$. It is clear that if $E$ is a compact and countable set and $E(\beta)$ is the last derivative of $E$, then $E(\beta) = E(\beta)$ and $E = \bigcup_{\beta \leq \alpha} E(\beta)$.

Example 1. Take on the $x$-axis the sets of points defined by: $E_1 = \{x; x = 1/n, n = 1, 2, \ldots \}$, $E_2 = \{x; x = 1/n + 1/m, m, n = 1, 2, \ldots \}$, $E_3 = (E_2 \setminus E_1) \cup \{0\}$. Then, the first coherence of $E_1$ is empty and the first derivative of $E_1$ consists of the point $x = 0$. The first coherence of the set $E_2$ consists of the point 0. The second coherence of $E_2$ is empty. The first derivative of $E_2$ is the set $E_1 \cup \{0\}$ and the second derivative of $E_2$ consists of the point 0.

We define now the rank $r_\alpha(B)$ of a point $p \in \overline{B}$, where $B$ is a countable set such that $\overline{B}$ is 0-dimensional.
DEFINITION. Let \( p \in \mathbb{B} \) where \( B \) is a countable set and \( \mathbb{B} \) is 0-dimensional. If \( p \in B_0 \) we define \( r_p(B) = 0 \). If there exists an \( \alpha \) such that \( p = \lim_{n \to \infty} p_n \) where \( p_n \in B_\alpha \) and \( p \) is not a limit point\(^{12}\) of \( B_{\alpha+1} \), we define \( r_p(B) = \alpha + 1 \).

If such an \( \alpha \) does not exist, then there exist an ordinal \( \alpha' \), a sequence \( \{ \alpha'_n \} \) of ordinals such that \( \alpha' = \alpha' \) and a sequence of points \( p_n \in B_\alpha \) such that \( p = \lim_{n \to \infty} p_n \) and \( p \) is not a limit point of \( B_\alpha \). In this case we define \( r_p(B) = \alpha' \).

EXAMPLE 2. If \( E_3 \) is the set defined in Example 1, the rank of the point 0 relative to \( E_3 \) is equal to 1.

Let now \( E_1 \) and \( E_2 \) be compact and countable sets, such that the \( \omega \)th derivative \( E_1^{(\omega)} \) of \( E_1 \) consists of the point \( p : E_1^{(\omega)} = (p) \) and the second derivative \( E_2^{(2)} \) of \( E_2 \) consists of the point \( q : E_2^{(2)} = (q) \). Put \( E_0 = E_1 \times (q) \cup (p) \times E_2 \) and \( B = [(p) \times E_2] \setminus (p, q) \). Then \( r_{(p, q)}(B) = 2 \) and \( r_{(p, q)}(E_2) = \omega \).

To define the family \( \mathcal{F} \) a few additional simple remarks are needed.

Since the order \( \alpha \) of a coherence is an invariant of homeomorphisms, it is easily seen that

(7) the rank \( r_p(B) \) is an invariant of homeomorphisms defined on \( \mathbb{B} \).

Take now the Cantor set \( C \) and let \( E \) be a compact and countable subset of \( C \) such that the \( \omega \)th derivative \( E^{(\omega)} \) of \( E \) consists of the point \( q : E^{(\omega)} = (q) \). Take the \( n \)th adherence \( E_n \) of \( E \), \( n = 1, 2, \cdots \) and choose from every \( E_n \) a point \( p_n \).

Since the order of an adherence is invariant under homeomorphisms we have that

(8) if \( h \) is any homeomorphism of \( E \) into itself, then \( h(p_n) = p_m \) for \( n \neq m \).

Let now \( D_n \) be the sequence of intervals in the plane defined by \( D_n = \{ (x, y) ; x = p_n, 0 \leq y \leq 1 \} \) \( n = 1, 2, \cdots \) and let \( \{ \alpha_n \} \) be a sequence of ordinals: \( 1 < \alpha_n < \Omega \). Choose in every \( D_n \) a countable and compact subset \( F_n \) such that \( \alpha_n \) be the order of the last derivative \( F_n^{(\alpha_n)} \) of \( F_n \) and that \( F_n^{(\alpha)} = (p_n) \). Then the set \( A = C \cup \bigcup_{n=1}^{\infty} F_n \) is compact (since the diameters of \( D_n \) are equal to \( 1/n \) and \( F_n \subset D_n \) and 0-dimensional. By the definition of \( F_n \) we have also

\[
\alpha_n > 1, \quad n = 1, 2, \cdots
\]

Now take in the plane an arbitrary bounded and isolated set \( I \)

\(^{12}\) A point \( x \) such that there exists a sequence \( \{ x_n \} \) of points \( x_n \) belonging to \( E \), \( x_n \neq x_m \) for \( n \neq m \) and such that \( x_n \to x \) is called a limit point of \( E \).

\(^{13}\) \( \times \) denotes the Cartesian product and \((p, q)\) is the point in the Cartesian product.
disjoint with $C$ such that $I^{(1)} = E$. Then the set $A_1 = C \cup \bigcup_{n=1}^{\infty} F_n \cup I$ is 0-dimensional and compact. Denoting the decomposition of $A_1$ according to the theorem of Cantor-Bendixson by $A_1 = P_1 \cup B_1$ with $P_1$ as perfect set, we obtain

$$P_1 = C \quad \text{and} \quad B_1 = \left( \bigcup_{n=1}^{\infty} F_n \cup I \right) \setminus E$$

and by the definition of $I$,

$$P_1 \cap B_1 = E.$$

Since $I$ is isolated there is also, by (9),

(10) \quad \rho_{p_n}(B_1) = \alpha_n > 1 \quad \text{and for every } p \in E \text{ and } p \neq p_m, \quad \rho_p(B_1) = 1.$$

If we take now any other sequence \( \{\beta_n\} \) of ordinals: \( 1 < \beta_n < \Omega \) and the same set $E$ and points $p_n$, we can construct, analogically as before, a 0-dimensional and compact set $A_2$ with the following properties:

If we denote the decomposition of $A_2$ according to the theorem of Cantor-Bendixson by $A_2 = P_2 \cup B_2$ with $P_2$ as perfect set, then $P_2 = C$ and

$$P_2 \cap B_2 = E.$$

Also

(10') \quad \rho_{p_n}(B_2) = \beta_n > 1 \quad \text{and for every } p \in E \text{ and } p \neq p_m, \quad \rho_p(B_2) = 1.$$

Now suppose that there exists a homeomorphism $h$ between $A_1$ and $A_2$: $h(A_1) = A_2$. Then we would have $h(P_1 \cap B_1) = P_2 \cap B_2$, i.e., $h(E) = E$. Hence by (8) there would be $h(p_n) \neq p_m$ for $n \neq m$. But, by (7), (10) and (10') there must be $h(p_n) = p_n$ and therefore by (7), $\alpha_n = \beta_n$ for every $n$. This shows that if the sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) are different, the sets $A_1$ and $A_2$ cannot be homeomorphic. But the power of all sequences \( \{\alpha_n\} \), \( 1 < \alpha_n < \Omega \) is $\aleph_1 = 2^\aleph_0$. Hence Theorem 2 holds.

Remark 3. In [7, p. 119], we introduced a function $\sigma_A(B)$ assigning to every 0-dimensional compact set $A$ an ordinal $< \Omega$. Using this function, it can be easily shown that the power of all topological types of compact uncountable subsets of the Cantor set is $\aleph_1$. (This can be also obtained from the result of Mazurkiewicz and Sierpinski, mentioned at the beginning of this paper.) Thus by the continuum hypothesis it is equal to $2^{\aleph_0}$, but we proved this fact without recourse to this hypothesis.

Note also that the fact that there exist $2^{\aleph_0}$ topological types of closed sets (not necessarily 0-dimensional) was stated in [6, p. 27].

3. In this section a proof of Theorem 3 is given. Two lemmas are also proved.
Lemma 2. Let $C_1$ and $C_2$ be two compact 0-dimensional sets and let $S_i \subseteq C_i$, $i = 1, 2$ such that $\text{Cl}(C_i \setminus S_i) = C_i$. Suppose that there exists a homeomorphism $h(C_i \setminus S_i) = C_2 \setminus S_2$ and let $p \in C_i \setminus S_i$ be a limit point of $S_i$. Then the point $h(p) = q$ is a limit point of $S_2$.

Proof. Suppose that $q$ is not a limit point of $S_2$. Since $C_2$ is 0-dimensional, there exists a closed and open (in $C_2$) neighbourhood $U \subseteq C_2$ of $q$ such that $U \cap S_2 = \emptyset$. $U$ being closed in $C_2$ it is compact; and since $h^{-1}$ is continuous $h^{-1}(U)$ is also a compact subset of $C_i \setminus S_i$. But $h^{-1}(U) \subseteq C_i \setminus S_i$ is also a neighbourhood of $p$, and since $\text{Cl}(C_i \setminus S_i) = C_i$ and $p$ is a limit point of $S_i$, there exists a point $p' \in S_i$ such that $p' \in h^{-1}(U)$, which is impossible.

As a trivial consequence of Lemma 2 we obtain the following:

Lemma 3. Let $C_1$ and $C_2$ be two perfect, 0-dimensional sets (containing more than one point) and let $S_i \subseteq C_i$, $i = 1, 2$ be two subsets of $C_i$ such that $S_i$ is denumerable. Suppose that there exists a homeomorphism $h(C_i \setminus S_i) = C_2 \setminus S_2$ and let $p \in C_i \setminus S_i$ be a limit point of $S_i$, then the point $h(p) = q$ is a limit point of $S_2$.

Indeed, since $S_i$ is denumerable we have $\text{Cl}(C_i \setminus S_i) = C_i$. The other assumptions of Lemma 2 being trivially satisfied it remains to apply this lemma.

Proof of Theorem 3. Since every subset of $C$ which is a $G_\delta$ set is defined by a sequence of open sets and the power of all open subsets of $C$ is $2^{\aleph_0}$, the power of all $G_\delta$ sets does not exceed $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$. Therefore it remains to construct a family of power $2^{\aleph_0}$ of $G_\delta$ sets which are dense in themselves and such that no two sets of this family are homeomorphic. We proceed to do this.

Take a perfect subset $P$ of the set $C$ which is nowhere dense in $C$. By Theorem 2 there exists a family $\mathcal{F}$ of power $2^{\aleph_0}$ of closed subsets of $P$ such that every two sets of $\mathcal{F}$ are not homeomorphic. Since $P$ is closed and nowhere dense in $C$ the sets of $\mathcal{F}$ are nowhere dense closed subsets of $C$. Thus for every set $F \subseteq \mathcal{F}$ there exists a sequence $S \subseteq C$ of points such that $S \subseteq C\setminus S$ and $S = F \cup S$. Now take two sets $F_1$ and $F_2$ of $\mathcal{F}$ and two sequences $S_1$ and $S_2$ of points such that $S_i \subseteq C$, $F_i \subseteq C \setminus S_i$ and $S_i = F_i \cup S_i$.

Consider the sets $C \setminus S_i$, $i = 1, 2$. We shall show that these sets are not homeomorphic. Indeed, suppose that there exists a homeomorphism $h(C \setminus S_i) = C \setminus S_2$. Since $S_i$ is denumerable and $C$ is perfect the assumptions of Lemma 3 hold for $C_1 = C_2 = C$. Thus, by $F_i \subseteq C \setminus S_i$ and $S_i = F_i \cup S_i$ every point $p$ of $F_1$ has an image $h(p)$ in $F_2$ and conversely.

$^{14}$ Evidently $P$ is homeomorphic to $C$. 
for every $q \in F_2$ there is $h^{-1}(q) \in F_1$. Hence by $h(C \setminus S_1) = C \setminus S_2$ there is $h(F_1) = F_2$ which is impossible by $F_i \in \mathfrak{F}$, $i = 1, 2$.

Thus we can correspond to every set $F \in \mathfrak{F}$ a set $C \setminus S$, where $S$ is denumerable, in such a way that the sets $C \setminus S_1$ and $C \setminus S_2$, corresponding to different sets $F_1$ and $F_2$ of $\mathfrak{F}$, are not homeomorphic. Since the power of $\mathfrak{F}$ is $2^{\aleph_0}$, the power of the family of corresponding sets of the form $C \setminus S$ is also $2^{\aleph_0}$. Since $S$ is denumerable the sets $C \setminus S$ are $G_\delta$ sets and since $C$ is perfect they are also dense in themselves. Hence Theorem 3 holds.

References


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